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M20580 L.A. and D.E. Tutorial
Quiz 4

1. Consider the invertible matrix
$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   (a) If P is the change of basis matrix from the basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$ to the standard basis of $\mathbb{R}^2$, what are the coordinate vectors of $\mathbf{v_1}$ and $\mathbf{v_2}$ with respect to the standard basis?

   (b) If P is the change of basis matrix from the standard basis of $\mathbb{R}^2$ to the basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$, what are the coordinate vectors of $\mathbf{v_1}$ and $\mathbf{v_2}$ with respect to the standard basis?

Solution:

(a) If P is the change of basis matrix from the basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$ to the standard basis of $\mathbb{R}^2$, then with respect to the standard basis $\mathbf{v_1} = \begin{bmatrix} a \\ c \end{bmatrix}$, and $\mathbf{v_2} = \begin{bmatrix} b \\ d \end{bmatrix}$.

(b) If P is the change of basis matrix from the standard basis of $\mathbb{R}^2$ to the basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$, then P^{-1} is the change of basis matrix from basis the $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$ to the standard basis of $\mathbb{R}^2$. Since $P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then with respect to the standard basis $\mathbf{v_1} = \begin{bmatrix} \frac{d}{ad-bc} \\ \frac{-c}{ad-bc} \end{bmatrix}$, and $\mathbf{v_2} = \begin{bmatrix} \frac{-b}{ad-bc} \\ \frac{a}{ad-bc} \end{bmatrix}$.

2. Let $p(x) = 1 - 2x,\ q(x) = x - x^2$, and $r(x) = -2 + 3x + x^2$ be polynomials in $\mathcal{P}_2$. Determine whether $s(x) = 3 - x - 5x^2$ is in span $\{p(x), q(x), r(x)\}$.

   Note: No credit for a *Yes/No* guess. You should clearly show the technique or reference relevant theorems.

Solution: The coordinate vectors of these polynomials with respect to the standard basis of $\mathcal{P}_2$ are

$$[p(x)]_{std} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad [q(x)]_{std} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad [r(x)]_{std} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad [s(x)]_{std} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}.$$

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The equation $ap(x) + bq(x) + cr(x) = s(x)$ in the unknown a, b, c, gives us a linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ -2 & 1 & 3 & -1 \\ 0 & -1 & 1 & -5 \end{array}\right].$$

This row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which tell us there are infinitely many solutions. So, $s(x)$ is in span $\{p(x), q(x), r(x)\}$.