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**M20580 L.A. and D.E. Tutorial**
**Quiz 5**

1. Given polynomials $p_1(x) = x^2 + x + 1$, $p_2(x) = 2x^2 - 1$, $p_3(x) = x^2 - x - 2$, determine if they are linearly dependent or linearly independent.

> ***Solution:*** Let's assign columns to the polynomials:
>
> $$[p_1(x)] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ [p_2(x)] = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \ [p_3(x)] = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$
>
> Now, concatenate the columns to form a matrix $A = [[p_1(x)] \vdots [p_2(x)] \vdots [p_3(x)]]$. If the matrix has a full rank, then the polynomials are linearly independent; otherwise, they are linearly dependent. So, let's row reduce $A$:
>
> $$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 - R1,\ R_3 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$
>
> We see that the rank is not maximal, hence the polynomials are linearly dependent.

2. Determine which of the following sets are subspaces of $\mathcal{P}_2$. If a set is not a subspace, provide a counterexample showing why; if it's a subspace, then justify why.

   1) $W_1 = \{p(x) \in \mathcal{P}_2 \mid xp'(x) = p(x) + 1\}$;
   2) $W_2 = \{p(x) \in \mathcal{P}_2 \mid p(0)p(1) = 0\}$;
   3) $W_3 = \{p(x) \in \mathcal{P}_2 \mid p(5) = 0\}$.

> ***Solution:*** The set $W_1$ does not contain the zero polynomial $p(x) = 0$, for it doesn't satisfy the defining condition of $W_1$, hence it's not a subspace. Next, $x \in W_2$ and $1 - x \in W_2$, but $x + (1 - x) = 1 \notin W_2$, for the constant polynomial 1 doesn't satisfy the defining condition of $W_2$. Lastly, $W_3$ is a subspace: if $p_1(x), p_2(x) \in W_3$, then $(p_1 + p_2)(5) = p_1(5) + p_2(5) = 0$; for any scalar $c$, $c \cdot p_1(5) = 0$. The last subspace can also be seen as the null space of the linear transformation $T(p(x)) = p(5)$.