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# M20580 L.A. and D.E.
# Quiz 7

1. For matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
   (a) What is the characteristic polynomial of A?
   (b) What are the eigenvalues of A?
   (c) What are the eigenvectors (up to scalars) with respect to each eigenvalues?.
   (d) Check if $A$ is diagonalizable. If $A$ is diagonalizable, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.
   (e) Use result from (d) to compute $A^{100}$.
   (you don't need to calculate the hundredth powers of numbers; for instance, leave $2^{100}$ without further calculating it)

***Solution:***

(a) The characteristic polynomial of A is:

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$

(b) $0 = (\lambda+2)(\lambda-5)$ give us $\lambda = -2$ and $\lambda = 5$

(c) • For $\lambda = -2$:

$$(A + 2I_2)\mathbf{x} = 0 \iff \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

and hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \; t \in \mathbb{R}.$$

• For $\lambda = 5$: using the same method, the eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; t \in \mathbb{R}.$$

(d) According to the theory of diagonalization, the following matrices

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

satisfy $A = PDP^{-1}$.

(e)

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \implies P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix};$$
$$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \implies D^{100} = \begin{bmatrix} (-2)^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Then

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \qquad \text{(100 times)} \\ &= PD(P^{-1}P)D(P^{-1}P)D\dots(P^{-1}P)DP^{-1} \\ &= PD^{100}P^{-1} \\ &= \frac{1}{7}\begin{bmatrix} (3)5^{100} + 2^{102} & 4 \cdot 5^{100} - 2^{102} \\ (3)5^{100} - (3)2^{100} & 3 \cdot 2^{100} + (4)5^{100} \end{bmatrix}. \end{aligned}$$