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M20580 L.A. and D.E. Tutorial
Worksheet 10

1. Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Solution: First we use Gram-Schmidt process to produce an orthogonal set.

$$v_1 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_3 &= x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2 \\ &= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{1}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

Then the orthonormal basis for col(A) is

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}.$$

$A = QR$ for some upper triangular matrix R, to find R we use the fact that Q has orthonormal columns, hence $Q^TQ = I$. Therefore $Q^TA = Q^TQR = IR = R$

$$R = Q^TA = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & -2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}.$$

$$A = QR = \begin{bmatrix} 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

2. Let $W = \text{span}\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$. Write a formula (as a matrix) for $\text{proj}_W(x)$, the orthogonal projection from $\mathbb{R}^3$ to W. This matrix, P, is called the standard matrix of the orthogonal projection onto the subspace W. Use this matrix to find the orthogonal projection of $v = \begin{bmatrix}1\\1\\1\end{bmatrix}$ onto W.

Hint: you will need an orthogonal basis for W and a formula for $\text{proj}_W(x)$ in terms of the orthogonal basis. So, we want a matrix P such that $\text{proj}_W(x) = P \cdot x$.

Solution: Recall that if W is a subspace of V, and $w_1, \dots w_k$ is an orthogonal basis of W, then:

$$\text{proj}_W(x) = \text{proj}_{w_1}(x) + \cdots + \text{proj}_{w_k}(x)$$

where for each vector v

$$\text{proj}_v(x) = \left(\frac{v \cdot x}{v \cdot v}\right) v.$$

Then first, we must find an orthogonal basis for W using Gram-Schmidt:

$$w_1 = x_1 = \begin{bmatrix}2\\0\\1\end{bmatrix},$$

$$w_2 = x_2 - \left(\frac{v_1 \cdot x_2}{w_1 \cdot w_1}\right) w_1 = \begin{bmatrix}2\\1\\1\end{bmatrix} - \frac{5}{5}\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

We have

$$\begin{aligned}
\text{proj}_W(x) &= \text{proj}_{\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right]}(x) + \text{proj}_{\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]}(x) \\
&= \frac{2x_1 + x_3}{5}\begin{bmatrix}2\\0\\1\end{bmatrix} + \frac{x_2}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix}4x_1 + 2x_3\\0\\2x_1 + x_3\end{bmatrix} + \begin{bmatrix}0\\x_2\\0\end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix}4x_1 + 2x_3\\5x_2\\2x_1 + x_3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4 & 0 & 2\\0 & 5 & 0\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}
\end{aligned}$$

Therefore,

$$P = \frac{1}{5}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

and

$$\mathrm{proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = P\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 1 \\ 3/5 \end{bmatrix}$$

3. For each of the following differential equations, first check if it is linear, then solve it using an appropriate method.

   (a) $\dfrac{dy}{dt} = ty^2 \cos t, \qquad y(0) = 1$

   (b) $t\dfrac{dy}{dt} = t^2 + y, \qquad y(1) = -2 \qquad \text{for } t > 0,$

   (c) $y' = 2ty + 3t^2 e^{t^2},$

   (d) $t^2 \dfrac{dy}{dt} + ty = 1$, assuming $t > 0$.

Solution:

(a) The DE is **not linear**. Using the method of separation of variables, we have:

$$\begin{aligned}
& \frac{dy}{dt} = ty^2 \cos t \\
\implies & \frac{dy}{y^2} = t \cos t \; dt \\
\implies & \int y^{-2} dy = \int t \cos t \; dt \\
& \qquad \text{Integration by parts:} \\
& \qquad \qquad u = t \implies du = dt \\
& \qquad \qquad dv = \cos t \; dt \implies v = \sin t \\
\implies & \frac{y^{-1}}{-1} = t \sin t - \int \sin t \; dt \\
\implies & -y^{-1} = t \sin t - (-\cos t) + C \\
\implies & -y^{-1} = t \sin t + \cos t + C.
\end{aligned}$$

Since $y(0) = 1$, we have

$$-1^{-1} = 0 \sin 0 + \cos 0 + C \implies \qquad C = -1^{-1} - \cos 0 = -1 - 1 = -2.$$

Therefore,

$$\begin{aligned}
& -y^{-1} = t \sin t + \cos t - 2 \\
\implies & y^{-1} = 2 - t \sin t - \cos t \\
\implies & \boxed{y = \frac{1}{2 - t \sin t - \cos t}}.
\end{aligned}$$

(b) The DE is **linear**. We have

$$\begin{aligned} t\frac{dy}{dt} &= t^2 + y \\ \implies t\frac{dy}{dt} - y &= t^2 \\ \implies \frac{dy}{dt} - t^{-1}\cdot y &= t \end{aligned}$$

Multiplying both sides of the equation by the following integrating factor

$$I = e^{\int -t^{-1}dt} = e^{-\ln|t|} = e^{-\ln t} \qquad (|t| = t \text{ as } t > 0)$$

$$I = t^{-1}. \qquad (\text{Note: } e^{-\ln x} \text{ is NOT } -x)$$

We then have

$$\begin{aligned} t^{-1}\frac{dy}{dt} - t^{-2}y &= 1 \\ \implies (t^{-1}y)' &= 1 \\ \implies t^{-1}y &= \int 1\cdot dt \\ \implies t^{-1}y &= t + C \\ \implies y &= t^2 + Ct. \end{aligned}$$

Since $y(1) = -2$, we have

$$-2 = 1^2 + C\cdot 1 \implies C = -2 - 1 = -3.$$

Therefore, $\boxed{y = t^2 - 3t}$.

(c) The DE is **linear**. We have

$$\begin{aligned} y' &= 2ty + 3t^2e^{t^2} \\ \implies y' - 2t\cdot y &= 3t^2e^{t^2}. \end{aligned}$$

We multiply both sides of the equation by the following integrating factor

$$I = e^{\int(-2t)dt} = e^{-t^2}.$$

We then have

$$\begin{aligned} y'e^{-t^2} - 2te^{-t^2}y &= 3t^2 \\ \implies (ye^{-t^2})' &= 3t^2 \\ \implies ye^{-t^2} &= \int 3t^2dt \\ \implies ye^{-t^2} &= t^3 + C \\ \boxed{y = e^{t^2}(t^3 + C)}, \end{aligned}$$

where C is an arbitrary constant.

(d) The DE is **linear**. We have

$$\begin{aligned} & t^2\frac{dy}{dt} + ty = 1 \\ \Longrightarrow\ & \frac{dy}{dt} + t^{-1}\cdot y = t^{-2}. \end{aligned}$$

We multiply both sides of the equation by the following integrating factor

$$I = e^{\int t^{-1}dt} = e^{\ln|t|} = |t| = t. \qquad (|t| = t \text{ for } t > 0)$$

We have

$$\begin{aligned} & t\frac{dy}{dt} + y = t^{-1} \\ \Longrightarrow\ & (ty)' = t^{-1} \\ \Longrightarrow\ & ty = \int t^{-1}dt \\ \Longrightarrow\ & ty = \ln|t| + C \\ \Longrightarrow\ & ty = \ln t + C \qquad (|t| = t \text{ for } t > 0) \\ & \boxed{y = t^{-1}\ln t + Ct^{-1}}. \end{aligned}$$

4. The population $p(t)$ at time t years of an animal grows according to a logistic growth model with intrinsic growth rate $0.4\,\mathrm{yr}^{-1}$ and environment carrying capacity 4 thousand. If harvesting is allowed at a rate of 0.3 thousand per year, sketch the graph of possible evolution of the population, find the phase portrait and classify all critical points.

   Hint: The evolution of $p(t)$ is given by the following differential equation:

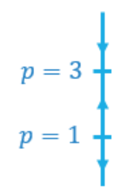
   $$\frac{dp}{dt} = 0.4p\left(1 - \frac{p}{4}\right) - 0.3$$

   The first step is to factor the RHS as a polynomial.

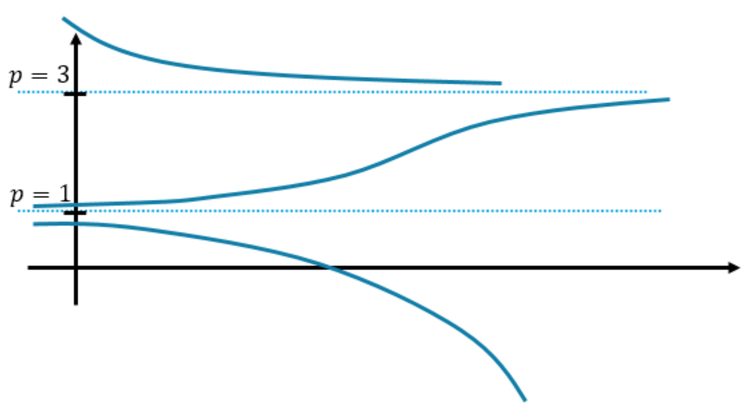
Solution: The RHS factors as

$$\frac{dp}{dt} = -0.1(p-1)(p-3).$$

Now we need to study the sign of $\frac{dp}{dt}$ depending on the value of p. If $p < 1$, then $\frac{dp}{dt} < 0$, which means that on the phase portrait we have an arrow down $\downarrow$ (the solution curves that start below $p = 1$ diverge from the line $p = 1$). Similarly, if $1 < p < 3$, then $\frac{dp}{dt} > 0$ (the solution curves diverge from $p = 1$ and converge to $p = 3$). Lastly, if $p > 3$, then $\frac{dp}{dt} < 0$ (the solutions converge to $p = 3$). Using this information, we can draw a phase portrait:



Now, using the phase portrait, we can draw the evolution of the population:



The critical point $p = 1$ is unstable and $p = 3$ is stable.

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5. Find an implicit solution to the IVP:

$$\sin(xy) + xy\cos(xy) + 2x + (x^2\cos(xy) + 2y)\frac{dy}{dx} = 0, \; y(4) = 0.$$

Solution: For a nonlinear equation this complicated, our only real hope is to see if the equation comes from an exact differential and then proceed by by partial integration. By multiplying both sides by dx we can arrive at the differential equation

$$M(x, y)dx + N(x, y)dy = 0,$$

$$\text{with } M(x, y) = \sin(xy) + xy\cos(xy) + 2x, \text{ and } N(x, y) = x^2\cos(xy) + 2y.$$

We now can check exactness:

$$M_y = 2x\cos(xy) - x^2y\sin(xy) = N_x.$$

Since M and N are everywhere differentiable, we can conclude there is a potential function $S(x, y)$ defined around $(4, 0)$ such that $S_x = M$ and $S_y = N$. Integrating partially, we find for some function $g(x)$,

$$S(x, y) = \int N(x, y)\; dy = x\sin(xy) + y^2 + g(x).$$

Then differentiating this with respect to x, we see

$$S_x = M(x, y) = \sin(xy) + xy\cos(xy) + g', \text{ so } g' = 2x, \text{ i.e. } g = x^2.$$

Therefore the general solution is

$$dS = 0, \text{ or } x\sin(xy) + x^2 + y^2 = C.$$

Now we use our initial condition, setting $y = 0$ and $x = 4$ to get $C = 16$; in other words the solution is

$$x\sin(xy) + x^2 + y^2 = 16.$$