Name: Date: 11/18/2021

M20580 L.A. and D.E. Tutorial
Worksheet 11

1. Determine whether the statements are true or false, and justify your answer.

   - If A is a $m \times n$ the QR factorization of A always exists.
   - Every orthogonal set of nonzero vectors are linearly independent.
   - If $Ax = b$ is a consistent linear system, then $A^T Ax = A^T b$ is also consistent, with the same solution.
   - If $Ax = b$ is an inconsistent linear system, then $A^T Ax = A^T b$ is also inconsistent.
   - For a subspace W of $\mathbb{R}^N$, $\text{proj}_W(x)$ is the best approximation to x in W.

Solution:

- False. It is a necessary additional hypothesis that the columns of A are linearly independent.
- True.
- True. A solution to $A^T Ax = A^T b$ is a least squares solution to $Ax = b$. If $Ax = b$ is consistent, then multiplying both sides on the left by A^T gives $A^T Ax = A^T b$. In other words, any solution to $Ax = b$ is also a solution to $A^T Ax = A^T b$.
- False. $A^T Ax = A^T b$ is always consistent. A solution is a least squares solution to $Ax = b$.
- True.

2. (a) Solve the separable differential equation: $dx + e^{4x}dy = 0$
   (b) Solve the linear differential equation: $y' + 3x^2y = x^2$.
   (c) Check if the differential equation is exact, then solve: $(2x-1)dx + (3y+7)dy = 0$

Solution:

(a) From $dy = -e^{-4x}dx$ we obtain $y = \frac{1}{4}e^{-4x} + c$.

(b) An integrating factor is $e^{\int 3x^2 dx} = e^{x^3}$ so that $\dfrac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$.

(c) Let $M = 2x-1$ and $N = 3y+7$ so that $M_y = 0 = N_x$. From $f_x = 2x-1$ we obtain $f = x^2 - x + h(y)$, $h'(y) = 3y+7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

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3. Solve the given initial-value problem

   (a) $\dfrac{dy}{dx} = x + 5y$, $y(0) = 3$

   (b) $xdx + (x^2y + 4y)dy = 0$, $y(4) = 0$.
   This ODE is not exact, but you can make it exact by first finding an integrating factor.

Solution:

(a) For $y' - 5y = x$, an integrating factor is $e^{\int -5dx} = e^{-5x}$ so that $\dfrac{d}{dx}e^{-5x}y = xe^{-5x}$ and

$$y = e^{5x}\int xe^{-5x}dx = -\frac{1}{5}x - \frac{1}{25} + ce^{5x}.$$

If $y(0) = 3$, then $c = \frac{76}{25}$ and $y = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}$.

(b) We note that $(M_y - N_x)/N = 2x/(4 + x^2)$, so an integrating factor is $e^{-2\int xdx/(4+x^2)} = 1/(4 + x^2)$.

Let $M = x/(4 + x^2)$and $N = (x^2y + 4y)/(4 + x^2) = y$, so that $M_y = 0 = N_x$. From $f_x = x/(4 + x^2)$ we obtain $f = \frac{1}{2}\ln(4 + x^2) + h(y)$, $h'(y) = y$, and $h(y) = \frac{1}{2}y^2$. A solution of the differential equation is

$$\frac{1}{2}\ln(4 + x^2) + \frac{1}{2}y^2 = c.$$

Multiplying both sides by 2 the last equation can be written as $e^{y^2}(x^2+4) = c_1$. Using the initial condition $y(4) = 0$ we see that $c_1 = 20$. A solution of the initial-value problem is $e^{y^2}(x^2 + 4) = 20$.

4. A tank originally has 100 liters of a brine with a concentration of 0.05 grams of salt per liter. Brine with concentration of 0.02 grams of salt per liter is pumped into the tank at a rate of 5 liters per second. The mixture is kept stirred and is pumped out at a rate of 4 liters per second. Find the amount of salt in the tank as a function of time.

Solution: Let $y(t)$ be the amount of salt in the tank at t minutes, then

$$\frac{dy}{dt} = \text{rate of incoming salt} - \text{rate of outgoing salt}.$$

and

$$\begin{aligned}
\text{rate of incoming salt} &= (\text{rate of incoming volume of brine}) \times (\text{incoming density}) \\
&= 5 \times 0.02 = 0.1 \\
\text{rate of outgoing salt} &= (\text{rate of outgoing volume of brine}) \times (\text{outgoing density}) \\
&= 4 \times \frac{y(t)}{100 + (5-4)t} = \frac{4y(t)}{100+t}.
\end{aligned}$$

so we obtain the 1st linear order DE

$$\frac{dy}{dt} = 0.1 - \frac{4}{100+t} y.$$

Solving this DE we obtain

$$y(t) = 0.02(100+t) + C(100+t)^{-4}$$

(Note: Don't be tempted to expand $(100+t)^4$). With the initial condition $y(0) = 100 \times 0.05 = 5$, we have $C = 3 \times 100^4 = 3 \times 10^8$, so

$$\boxed{y(t) = 0.02(100+t) + \frac{3 \times 10^8}{(100+t)^4}} \quad (g).$$