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# M20580 L.A. and D.E. Tutorial
## Worksheet 2

1. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

   (a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

   (b) A homogeneous linear system in $n \geq 1$ unknowns whose corresponding augmented matrix has a reduced row echelon form with $r$ leading 1's has $n - r$ free variables.

   (c) If a homogeneous linear system of $n \geq 1$ equations in $n$ unknowns has a corresponding augmented matrix with a reduced row echelon form containing $n$ leading 1's, then the linear system has only the trivial solution.

   (d) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

   (e) If a consistent linear system has more unknowns than equations, then it must have infinitely many solutions.

***Solution:***

(a) True. If a matrix is in reduced row echelon form, then in particular it is in row echelon form.

(b) True. We have an augmented matrix with $r$ pivot columns. Therefore, there are $n - r$ columns which are not pivots, hence $n - r$ free variables.

(c) True. There are $n$ pivot columns and zero free variables. So, there is a unique solution which must be the trivial one.

(d) False. Consider the augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. This matrix is in reduced row echelon form, but it has just one solution, the trivial one.

(e) True. If there are $n$ unknowns and $m$ equations where $n > m$, then there will be at most $m$ pivot columns and therefore at least $n - m > 0$ free variables. So, there will be infinitely many solutions.

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$ Compute $A^T A + 6B^{-1}$.

***Solution:***

$$\begin{aligned} A^T A + 6B^{-1} &= \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} + \frac{6}{2} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 14 \\ 14 & 21 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 26 \\ 14 & 24 \end{bmatrix} \end{aligned}$$

3. Find the inverses of the following matrices if it exists

(a) $\begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -8 & -4 \\ 6 & 3 \end{bmatrix}$

***Solution:*** We recall that for a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad - bc \neq 0$, then the inverse of $A$ is given by $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, then $A$ does not have an inverse (not invertible).

(a) $3 \times 4 - 2 \times 7 = -2 \neq 0$, so the inverse is

$$\frac{1}{-2} \begin{bmatrix} 3 & -2 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{7}{2} & -2 \end{bmatrix}$$

(b) $(-8) \times 3 - (-4) \times 6 = 0$ so the matrix is not invertible.

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

   Is $T$ is a linear transformation? Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation $T$.

***Solution:*** Let $A$ be an $m \times n$ matrix. Then the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad (\text{for } \mathbf{x} \text{ in } \mathbb{R}^n) \qquad\qquad (\text{Theorem 3.30: Poole's book})$$

is a linear transformation.

So $T$ is a linear transformation.

We then have

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 11 \end{bmatrix}.$$

5. (a) Let $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$.

Construct a $2 \times 2$ matrix $B$ such that $AB$ is the zero matrix. Use two different **nonzero** columns for $B$.

***Solution:*** We note that if we write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{b}_1, \mathbf{b}_2$ are two column vectors in $\mathbb{R}^2$, then $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$ (See, e.g., textbook Theorem 3.31 and its proof). Hence, $AB$ is a zero matrix if and only if $A\mathbf{b}_1 = A\mathbf{b}_2 = \mathbf{0}$, so we need to find two (nonzero) solutions to the system $A\mathbf{x} = \mathbf{0}$.

Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The RREF of $A$ is $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, so

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x_1 - 3x_2 & = 0 \\ 0x_1 + 0x_2 & = 0 \quad \text{(trivial)} \end{cases} \iff x_1 = 3x_2.$$

We can choose $x_2 = 1$ and then $x_2 = -1$, which gives $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, so one choice for a matrix $B$ that satisfies $AB = 0$ is $B = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

Find the conditions on $a, b, c$, and $d$ such that $A$ commutes with both $B$ and $C$, that is, $AB = BA$ and $AC = CA$.

***Solution:*** We can work out to see that

$$AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}, \qquad BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

so by comparing entry-by-entry, we see that $AB = BA$ if and only if $b = 0$ and $c = 0$.

Likewise, we have

$$AC = \begin{bmatrix} a & -3a + b \\ c & -3c + d \end{bmatrix}, \qquad CA = \begin{bmatrix} a - 3c & b - 3d \\ c & d \end{bmatrix},$$

so that $AC = CA$ if and only if $a = a - 3c$ and $-3a + b = b - 3d$ and $-3c + d = d$. Solving these three conditions simultaneously, we obtain $c = 0$ and $a = d$.

Hence, $A$ commutes with both $B$ and $C$ if $b = c = 0$ and $a = d$, where $a$ can be any number.