Name: Date: 09/09/2021

M20580 L.A. and D.E. Tutorial
Worksheet 3

1. Use Guass-Jordan method to find the inverse of the given matrix (if it exists):

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1}$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 0 & 1 & 3 & -1 & -2 \end{array}\right]$

$$\Rightarrow A^{-1} = \begin{bmatrix} 4 & -1 & -4 \\ -3 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix}$$

2. Find the standard matrices of the following linear transformations:

   a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+7y \\ y-4x \end{bmatrix}$

   b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-z \\ z-x+y \\ 0 \\ 5y+3z \end{bmatrix}$

Solution:

Recall that the standard matrix A for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, is the matrix satisfying $A\mathbf{u} = T(\mathbf{u})$, where $\mathbf{u} \in \mathbb{R}^n$.

a) Since $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+7y \\ y-4x \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 3 & 7 \\ -4 & 1 \end{bmatrix}$

b) Since $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-z \\ z-x+y \\ 0 \\ 5y+3z \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & 3 \end{bmatrix}$

3. (a) Is $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ -y \end{bmatrix}$ a linear transformation?

If not, which component of this transformation is "non-linear"? (Here the two components of this map are x^2 and $-y$.)

(b) Given a linear transformation T with $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$,

what is $T\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

Solution: (a) This is not a linear transformation.

For example $2T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \neq T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

The non-linear component is x^2. It also follows from the previous example that T_1(the first component) gives $2T_1(1) = 2 \neq T_1(2) = 4$. However, $T_2(y) = -y$ is a typical linear transformation.

(b) $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$

4. (a) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation. If $T(\mathbf{u}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, T(\mathbf{w}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$, where $\mathbf{x} = 4\mathbf{u} + 3\mathbf{v} + 2\mathbf{w}$.

Solution: We recall two properties of a linear transformation T: for any two vectors of appropriate size $\mathbf{u}, \mathbf{v}$ and any scalar c, we have

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Now, we continually use these two properties to compute $T(\mathbf{x})$. We have

$$\begin{aligned} T(\mathbf{x}) &= T(\mathbf{4u} + 3\mathbf{v} + 2\mathbf{w}) = T(4\mathbf{u}) + T(3\mathbf{v}) + T(2\mathbf{w}) && \text{(property (1))} \\ &= 4T(\mathbf{u}) + 3T(\mathbf{v}) + 2T(\mathbf{w}) && \text{(property (2))} \\ T(\mathbf{x}) &= 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \end{bmatrix}. \end{aligned}$$

(b) Continuing part (a), if we know $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x}$ in $\mathbb{R}^3$.

Solution: If we write $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for any vector in $\mathbb{R}^3$, we have

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

so using the same argument as in (b), we have

$$\begin{aligned} T(\mathbf{x}) &= aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) \\ &= a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a + b + 2c \\ 1a + 3b + 2c \end{bmatrix} \\ T(\mathbf{x}) &= \begin{bmatrix} \langle 2, 1, 2 \rangle \cdot \langle a, b, c \rangle \\ \langle 1, 3, 2 \rangle \cdot \langle a, b, c \rangle \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{x}. \end{aligned}$$

Therefore, $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$.

5. Given a matrix A, $Null(A)$ is the collection of all vectors v such that Av is trivial (all entries of Av are zero), $Row(A)$ is the collection of all possible linear combinations of rows of A, and $Col(A)$ is the collection of all possible linear combinations of columns of A.

   Suppose $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$.

   (a) Is $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ in $Null(A)$?

   (b) Is $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix}$ in $Row(A)$?

Solution: (a) No. Because $A \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 2 \end{bmatrix}$ is not trivial.

(b) Yes.

Suppose $xR_1 + yR_2 + zR_3 = \begin{bmatrix} 6 & 4 & 2 \end{bmatrix}$, we get a linear system.

Solve it for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then we get a unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix} = 2R_1 + R_2 + R_3$