Name: Date: 09/16/2021

M20580 L.A. and D.E. Tutorial
Worksheet 4

1. Given the matrix $A = \begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{bmatrix}$, find a basis for Row($A$), Col($A$) and Null($A$).

   Hint: the first step is to row reduce A. Then, the non-zero rows will form a basis of Row(A), and pivots will indicate which columns of A form a basis of Col(A) (but we do not pick columns of a REF of A for a basis of Col(A)!). For Null(A), augment A by zero and solve the resulting system.

Solution: Row reduce A:

$$\begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0. \end{bmatrix}$$

We see that a basis for Row(A) is $\{\begin{bmatrix} 1 & -1 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 5 \end{bmatrix}\}$; a basis for Col($A$) can be chosen as the first and the third columns of A: $\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\}$. For the null space, augment A by zero and solve

$$\begin{cases} x_1 - x_2 - 2x_3 + 3x_4 = 0 \\ x_3 = 5x_4 \end{cases}$$

The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Therefore, a basis for Null(A) can be chosen as

$$\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 5 \\ 1 \end{bmatrix}\}.$$

2. Given the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ and letting $\mathbf{v}$ denote the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, write $\mathbf{v}$ as a linear combination of the given vectors. Does the given set span $\mathbb{R}^3$? Is it a basis for $\mathbb{R}^3$?

Solution:

We have to try to solve the linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right].$$

Therefore $\mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$. Since the matrix whose columns were our starting vectors has the identity as RREF, we know it is a basis, in particular a spanning set, for $\mathbb{R}^3$.

3. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for a vector space V. Also suppose that the change-of-basis matrix **from** $\mathcal{B}$ **to** $\mathcal{C}$ is given as

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$

For $\mathbf{v} = \mathbf{b}_1 - 3\mathbf{b}_2$, what is $[\mathbf{v}]_{\mathcal{C}}$, the $\mathcal{C}$-coordinate for $\mathbf{v}$?

Solution:

$\mathbf{v} = \mathbf{b}_1 - 3\mathbf{b}_2$ means that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

So

$$[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

4. Find $\mathcal{C}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ from Question 3.

Solution: The basis $\mathcal{C}$ in terms of the standard basis $\mathcal{E}$ comprises the columns of $P_{\mathcal{E}\leftarrow\mathcal{C}}$, which can be expressed as

$$P_{\mathcal{E}\leftarrow\mathcal{C}} = P_{\mathcal{E}\leftarrow\mathcal{B}} P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{E}\leftarrow\mathcal{B}} (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$$

A computation yields

$$P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}.$$

Thus $\mathcal{C} = \left\{ \begin{bmatrix} -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$.

5. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}. \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, find $\mathbf{x}$ (its coordinate representation in the standard basis).

Solution:
$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ means that the coordinates of $\mathbf{x}$ relative to the $\mathcal{B}$ basis is $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, so

$$\mathbf{x} = -1 \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}.$$