Name: Date: 09/23/2021

M20580 L.A. and D.E. Tutorial
Worksheet 5

1. Determine whether the statements are true or false and justify your answer.

   (a) A set containing a single vector is linearly independent.

   (b) The set of vectors $\{v, kv\}$ is linearly dependent for every scalar k.

   (c) Every linearly dependent set contains the zero vector.

   (d) The span of $v_1, \ldots, v_n$ is the column space of the matrix whose column vectors are $v_1, \ldots, v_n$.

   (e) The column space of a matrix A is the set of solutions of $Ax = b$.

   (f) The system $Ax = b$ is inconsistent if and only if b is not in the column space of A.

Solution:

(a) F: $\{0\}$ is linearly dependent. Any set containing a single nonzero vector is linearly independent.

(b) T: $-kv + kv = 0$.

(c) F: $\{v, kv\}$ where $v \neq 0$ and $k \neq 0$.

(d) T: by definition.

(e) F: The column space of an $m \times n$ matrix A is the span of the columns. Equivalently, it is the set of all vectors b in $\mathbb{R}^m$ such that $Ax = b$ is consistent.

(f) T: If $Ax = b$ is inconsistent, then there does not exist a vector x such that $Ax = b$. Therefore, b is not a linear combination of the columns of A, hence not in the column space of A.

If b is not in the column space of A, then b is not a linear combination of the columns of A. So, there does not exist a vector x such that $Ax = b$, hence $Ax = b$ is inconsistent.

2. Show that the vectors $v_1 = (1, 0, -1)$, $v_2 = (1, 2, 1)$, $v_3 = (0, -3, 2)$ form a basis of $\mathbb{R}^3$.

Solution: The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So, v_1, v_2, v_3 are linearly independent and span all of $\mathbb{R}^3$.

3. Consider

$$\mathbf{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Find the coordinate vectors $[\mathbf{x}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{C}$.

(b) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

(c) Use $P_{\mathcal{C}\leftarrow\mathcal{B}}$ to compute $[\mathbf{x}]_\mathcal{C}$.

(d) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

(e) Use $P_{\mathcal{B}\leftarrow\mathcal{C}}$ to compute $[\mathbf{x}]_\mathcal{B}$.

Solution:

(a) To find $[\mathbf{x}]_\mathcal{B}$, we need to find scalars a and b such that $\begin{bmatrix} 4 \\ -2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This will give us that $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} a \\ b \end{bmatrix}$. This translates to solving the augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & -2 \end{array}\right]$. We find that $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

Similary, we find that $[\mathbf{x}]_\mathcal{C} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(b) The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$. In other words, the first column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is $[\mathbf{b}_1]_\mathcal{C}$, and the second column is $[\mathbf{b}_2]_\mathcal{C}$. Similar to part (a), we find that $[\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $[\mathbf{b}_2]_\mathcal{C} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(c) $[\mathbf{x}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(d) The columns of $P_{\mathcal{B}\leftarrow\mathcal{C}}$ are the coordinate vectors of $\mathcal{C}$ with respect to $\mathcal{B}$. In other words, the first column of $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is $[\mathbf{c}_1]_\mathcal{B}$, and the second column is $[\mathbf{c}_2]_\mathcal{B}$. Similar to part (a), we find that $[\mathbf{c}_1]_\mathcal{B} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $[\mathbf{c}_2]_\mathcal{B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(e) $[\mathbf{x}]_\mathcal{B} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_\mathcal{C} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

4. Consider $A = \begin{bmatrix} 1 & 1 & -8 \\ 0 & 2 & 1 \\ 1 & -1 & -9 \end{bmatrix}$. Give bases for $\text{row}(A), \text{col}(A), \text{null}(A)$.

Solution: $\text{REF}(A) = \begin{bmatrix} 1 & 1 & -8 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So, a basis for $\text{row}(A)$ is $\{\begin{bmatrix} 1 & 1 & -8 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}\}$; a basis for $\text{col}(A)$ is $\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\}$.

To describe the null space, it may be easier to first put the matrix in RREF and augment with zeros. We get $\left[\begin{array}{ccc|c} 1 & 0 & -17/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This gives $x_1 - \frac{17}{2}x_3 = 0$ and $x_2 + \frac{1}{2}x_3 = 0$. Then $x_1 = \frac{17}{2}x_3$ and $x_2 = -\frac{1}{2}x_3$. So, a basis for $\text{null}(A)$ is $\{\begin{bmatrix} 17/2 \\ -1/2 \\ 1 \end{bmatrix}\}$.

If we multiply this basis vector by 2, a nicer basis for $\text{null}(A)$ is $\{\begin{bmatrix} 17 \\ -1 \\ 2 \end{bmatrix}\}$.

5. Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

Solution: If three vectors in $\mathbb{R}^3$ are linearly dependent, then at least one of them must be in the span of the others. Since we want any two of these vectors to be linearly independent, let us start with two linearly independent vectors. Two such vectors are $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$. These two vectors span the plane $z = 0$ in $\mathbb{R}^3$. Now, we want to choose one more vector, v_3, such that v_1, v_2, v_3 are linearly dependent, but any two of these vectors are linearly independent. Equivalently, we want to choose v_3 so that v_3 is in the span of $\{v_1, v_2\}$, but v_3 is not the span of $\{v_1\}$ and not in the span of $\{v_2\}$. One such vector is $v_1 + v_2 = (1, 1, 0)$.

Therefore, one solution is $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$.