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M20580 L.A. and D.E. Tutorial
Worksheet 6

1. All of the following sets with their operations are not vector spaces. State at least one axiom of vector spaces that does not hold in each case and justify your answer with concrete examples:

   (a) The set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ with usual vector addition and scalar multiplication.

   (b) $\mathbb{R}^2$, with the usual scalar multiplication but addition is defined by

   $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix}.$$

   (c) The set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad = 0$ with usual matrix operations.

Solution:

(a) We have $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector of the proposed form, but $k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$ is not in the considered set if $k < 0$, so this set is not closed under scalar multiplication.

(b) Consider vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ with the scalar 2 we have

$$2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = 2\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

while

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \end{bmatrix}$$

so distributivity does not hold.

(c) The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in the considered set, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not, so this set is not closed under vector addition.

2. Determine whether the statements are true or false and justify your answer.

   (a) If $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all vectors v_1 and v_2 in V and all scalars c_1 and c_2 , then T is a linear transformation.

   (b) There is exactly one linear transformation $T : V \to W$ for which $T(u+v) = T(u-v)$ for all vectors u and v in V.

   (c) Let $\{v_1, \dots, v_n\}$ be a basis for a vector space V. If $T(v_1) = T(v_2) = \cdots = T(v_n) = 0$, then T is the zero transformation.

   (d) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and if $[T]_{B\to B} = I_n$ with respect to some basis B of $\mathbb{R}^n$, then T is the identity transformation.

   (e) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and if $[T]_{B\to C} = I_n$ with respect to two different bases B and C of $\mathbb{R}^n$, then T is the identity transformation.

Solution:

(a) True. This follows from the definition of linear transformation. Since $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv) = cT(v)$, we have $T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

(b) True. If $T(u+v) = T(u-v)$, then $T(u)+T(v) = T(u)-T(v)$. So, $T(v) = -T(v)$ hence $2T(v) = 0$. So, $T(v) = 0$ for all vectors $v \in V$. Therefore, T is the zero transformation.

(c) True. In general, a linear transformation is determined by where it sends the basis vectors. Any vector $v \in V$ can be written as $c_1v_1 + c_2v_2 + \cdots + c_nv_n$. Applying T and using the definition of linear transformation, we have $T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0$ since each each $T(v_i) = 0$. So, T is the zero transformation.

(d) True. If $[T]_{B\to B} = I_n$ then the transformation does nothing to each coordinate vector relative to the basis B. In other words, letting $B = \{v_1, \dots, v_n\}$, we have $T(v_1) = v_1, T(v_2) = v_2, \dots, T(v_n) = v_n$. Any vector $v \in \mathbb{R}^n$ can be written as $v = c_1v_1 + \cdots + c_nv_n$, and we have $T(v) = T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n) = c_1v_1 + \cdots + c_nv_n = v$. So, T is the identity transformation.

(e) False. Let $n = 1$ and choose the bases $B = \{1\}$(i.e., standard basis) and $C = \{2\}$ for $\mathbb{R}$. The transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $T(1) = 2$ is not the identity, but the matrix relative to $\{1\}$ and $\{2\}$ is $[T]_{B\to C} = [1]$ since $T(1) = 2 = 1 \cdot 2$.

3. In each of the following, V is a vector space and W is a subset of V. Determine if W is a subspace of V. Justify your answer.

   (a) $V = \mathbb{R}^3$, and $W = \left\{ \begin{bmatrix} a \\ 0 \\ -2a \end{bmatrix} \mid a \in \mathbb{R} \right\}$,

   (b) $V = \mathcal{P}_3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$, and $W = \{a_1 + b_1x + c_1x^2 + d_1x^3 \mid a_1, b_1, c_1, d_1 \in \mathbb{R} \text{ and } a_1 > b_1\}$,

   (c) $V = \mathcal{M}_{2\times 2}(\mathbb{R})$, and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } ad \geq 0 \right\}$.

Solution:

(a) $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$ which is a subspace of $V = \mathbb{R}^3$. (See, e.g., Theorem 6.3 Poole's book).

(b) The polynomial $p(x) = 2 + x$ is in W, but $(-1)p(x) = -2 - x$ has $-2 < -1$ is not in W. Hence W is not closed under scalar multiplication, and is not a subspace of V.

(c) The matrices, $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ are in W, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is not. Since W is not closed under vector addition, W is not a subspace of V.

4. Recall that for any two polynomials $p = p(x), q = q(x)$ and any scalar λ, we define $p+q$ to be the polynomial that satisfies

$$(p+q)(x) = p(x) + q(x)$$

and the polynomial λp to be

$$(\lambda p)(x) = \lambda \cdot p(x).$$

Define a transformation $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find the kernel of T and describe the range of T.

Solution:

(a) Let $p = p(x)$ and $q = q(x)$ be two elements in $\mathcal{P}_2$. We have

$$T(p) + T(q) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix}$$

and

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix},$$

so $T(p+q) = T(p) + T(q)$. Moreover, for any scalar $\lambda \in \mathbb{R}$,

$$T(\lambda p) = \begin{bmatrix} (\lambda p)(0) \\ (\lambda p)(1) \end{bmatrix} = \begin{bmatrix} \lambda \cdot p(0) \\ \lambda \cdot p(1) \end{bmatrix} = \lambda \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \lambda T(p)$$

so T satisfies is a linear transformation.

(b) Writing $p(x) = a + bx + cx^2$, we have $T(p)$ is the zero vector if and only if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

This gives $a = a+b+c = 0$, which gives $b = -c$ and then $(a,b,c) = (0,t,-t)$, $t \in \mathbb{R}$.
Hence, $ker(T) = \{tx - tx^2 \mid t \in \mathbb{R}\}$.

Now, if $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2$ is in the range of T, then for some $p(x) = a + bx + cx^2$, we have

$$\begin{bmatrix} m \\ n \end{bmatrix} = T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

So $a = m$ and $n = a + b + c = m + b + c$. Letting $b = -m$ and $c = n$, we have $T(m - mx + nx^2) = \begin{bmatrix} m \\ n \end{bmatrix}$. Hence the range of T is the whole $\mathbb{R}^2$

5. $V = \mathcal{P}_3$ is a vector space, and $S = \{x^3 - 2x^2 + 3x - 1, 2x^3 + x^2 + 3x - 2\}$ is a subset of V. Determine if the given vector $v = -2x^3 - 11x^2 + 3x + 2$ of V is in span(S).

   If it is, find an explicit representation of v as a linear combination of the vectors in S.

Solution: The equation we are solving is

$$\begin{aligned} -2x^3 - 11x^2 + 3x + 2 &= a(x^3 - 2x^2 + 3x - 1) + b(2x^3 + x^2 + 3x - 2) \\ &= (a + 2b)x^3 + (-2a + b)x^2 + (3a + 3b)x + (-a - 2b) \end{aligned}$$

This is equivalent to the following system linear equations (by equating coefficients of same x-power term):

$$\begin{cases} a + 2b &= -2, \\ -2a + b &= -11, \\ 3a + 3b &= 3, \\ -a - 2b &= 2. \end{cases}$$

We have

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 5 & -15 \\ 0 & -3 & 9 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

The system has solution $a = 4$ and $b = -3$, so that

$$-2x^3 - 11x^2 + 3x + 2 = 4(x^3 - 2x^2 + 3x - 1) - 3(2x^3 + x^2 + 3x - 2).$$

Thus v is in span(S).