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M20580 L.A. and D.E. Tutorial
Worksheet 7

1. Let $\mathcal{E} := \{e^{2x}, e^{-2x}\}$ and let $M := \operatorname{span}\mathcal{E}$ be a space spanned by $\mathcal{E}$. Consider another basis $\mathcal{B} := \{\sinh(2x), \cosh(2x)\}$ of M, where

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}, \quad \cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}.$$

   a) Find the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$.

   b) Find the standard matrices $[D]_{\mathcal{E}}$ and $[D]_{\mathcal{B}}$ of the linear transformation $D : M \to M$ given by $D(y(x)) = y''(x)$.

Solution: Let us pick $\mathcal{E}$ as the standard basis of M. In terms of $\mathcal{E}$, the basis $\mathcal{B}$ is

$$\mathcal{B} = \{[\sinh(2x)]_{\mathcal{E}},\ [\cosh(2x)]_{\mathcal{E}}\} = \{\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\}.$$

Accordingly, the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$ is simply a concatenation of the latter columns:

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix};$$

its inverse is

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = 2\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Now, the operator D acts upon e^{2x} and e^{-2x} as

$$D(e^{2x}) = 4e^{2x}, \quad D(e^{-2x}) = 4e^{-2x},$$

hence

$$[D]_{\mathcal{E}} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Therefore,

$$[D]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

A more direct way of computing $[D]_{\mathcal{B}}$ would be to notice that $D(\sinh(2x)) = 4\sinh(2x)$ and $D(\cosh(2x)) = 4\cosh(2x)$, which gives the columns of $[D]_{\mathcal{B}}$.

2.

a) Let $S : P_2 \to P_3$ be the integration linear transformation, i.e.,

$$S(a_0 + a_1x + a_2x^2) = \int_0^x (a_0 + a_1t + a_2t^2)\ dt = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3.$$

Find the matrix of S with respect to the standard bases for P_2 and P_3.

b) Find the matrix of the differentiation linear transformation $D : P_3 \to P_2$, i.e.,

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = \frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$$

with respect to the standard bases for P_3 and P_2.

c) What is the matrix of the composition $D \circ S$, i.e., $\left[\frac{d}{dx}\int_0^x\right]$?

d) Find a basis for the null space of S and a basis for the null space of D.

e) Find a basis for the column space of S and a basis for the column space of D.

Solution:

a) $[S] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$

b) $[D] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

c) $[D][S] = I_3$.

d) There are no free variables in the matrix of S. This tells us immediately that the null of S is consists of just the zero vector.

In the matrix of D, we can see that there is one free variable (the first variable). So, a basis for the null of D is just the vector $(1, 0, 0, 0)$, which corresponds the constant polynomial 1.

e) The matrices of S and D give us immediately a basis for their column spaces. For S, we can take all three columns as a basis. For D, we can take columns $2, 3, 4$ as a basis.

3. Let $T : M_{2\times 2} \to M_{2\times 2}$ be a linear transformation defined by

$$T(A) := AW_0 - W_0A, \quad \text{where} \quad W_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

a) Find the matrix $[T]_{\mathcal{E}}$ of T in the standard basis

$$\mathcal{E} = \left\{ E_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_3 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_4 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

b) Consider a basis $\mathcal{B}$ in $M_{2\times 2}$ given by

$$\mathcal{B} := \left\{ B_1 := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},\ B_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_3 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_4 := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right\}.$$

Compute $[T]_{\mathcal{B}}$.

c) Use $[T]_{\mathcal{B}}$ to compute the null space of T.

Hint: for part b), it would be easier if instead of using the formula $[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[T]_{\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}}$ you use $[T]_{\mathcal{B}} = [[T(B_1)]_{\mathcal{B}} \vdots [T(B_2)]_{\mathcal{B}} \vdots [T(B_3)]_{\mathcal{B}} \vdots [T(B_4)]_{\mathcal{B}}]$. I.e., try acting by T upon the matrices from $\mathcal{B}$, see how the outputs can be expressed as linear combinations of $\mathcal{B}$, and collect the coefficients into the columns of $[T]_{\mathcal{B}}$. Once you learn a bit more about linear algebra, you'll be able to see that $\mathcal{B}$ is a so-called *eigen-basis* of T.

Solution: The action of T upon $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$T(A) = \begin{bmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{bmatrix} - \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = \begin{bmatrix} a_{12} - a_{21} & a_{11} - a_{22} \\ a_{22} - a_{11} & a_{21} - a_{12} \end{bmatrix}.$$

Therefore, the matrix of T in $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Now, let's find $[T]_{\mathcal{B}}$ via $[T]_{\mathcal{B}} = [[T(B_1)]_{\mathcal{B}} \vdots [T(B_2)]_{\mathcal{B}} \vdots [T(B_3)]_{\mathcal{B}} \vdots [T(B_4)]_{\mathcal{B}}]$:

$$T(B_1) = -2B_1, \quad T(B_2) = 0 \cdot B_2, \quad T(B_3) = 0 \cdot B_3, \quad T(B_4) = 2B_4.$$

Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Now, the null space of T can be easily read off from $[T]_{\mathcal{B}}$:

$$\text{null}(T) = \text{span}\{B_2, B_3\}.$$

4. Define a linear transformation $T : \mathcal{P}_2 \to \mathbb{R}^2$ via
$$T(p(x)) = \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix}.$$
Let $\mathcal{B} = \{1,\ 1+x,\ 1+x+x^2\}$ be another basis for $\mathcal{P}_2$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ be another basis for $\mathbb{R}^2$. Compute $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

Hint: you may want to start with $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = [[T(1)]_{\mathcal{C}} \vdots [T(1+x)]_{\mathcal{C}} \vdots [T(1+x+x^2)]_{\mathcal{C}}]$; for this approach, you will also need to compute $P_{\mathcal{C}\leftarrow\mathcal{E}}$. Another, more computationally demanding way would be to use the formula $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}}[T]_{\mathcal{E}\leftarrow\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}}$.

Solution: The matrix $P_{\mathcal{C}\leftarrow\mathcal{E}}$ is given by
$$P_{\mathcal{C}\leftarrow\mathcal{E}} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix};$$
so,
$$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix};$$
$$[T(1+x)]_{\mathcal{C}} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix};$$
$$[T(1+x+x^2)]_{\mathcal{C}} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -13 \end{bmatrix}.$$
Thus
$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 6 & 8 \\ -3 & -10 & -13 \end{bmatrix}.$$