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M20580 L.A. and D.E. Tutorial
Worksheet 8

1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 3 & 1 & -6 \end{bmatrix}$.

   (a) Compute the first column of the cofactor matrix associated to A.

   (b) Using part (a) compute the determinant of A.

Solution:

(a) We compute C_{11} C_{21} and C_{31}.

$C_{11} = (-1)^{1+1}\det\left(\begin{bmatrix} 0 & -3 \\ 1 & -6 \end{bmatrix}\right) = (-1)^{1+1}((0)(-6) - (-3)(1)) = 3.$

$C_{21} = (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -6 \end{bmatrix}\right) = (-1)^{2+1}((1)(-6) - (1)(1)) = 7.$

$C_{31} = (-1)^{3+1}\det\left(\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}\right) = (-1)^{3+1}((1)(-3) - (1)(0)) = -3.$

(b) Using expansion along the first column, the determinant is $0C_{11}+1C_{21}+3C_{31} = 0(3) + 1(7) + 3(-3) = -2$.

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2. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

(a) What is the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

(b) If $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{C} = \{\beta_1, \beta_2\}$, where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0), \quad \beta_1 = (0, 1), \quad \beta_2 = (1, 0),$$

what is the matrix of $[T]_{B \to C}$.

Solution:

(a) Since $T(1, 0, 0) = (1, -1)$, $T(0, 1, 0) = (1, 0)$, $T(0, 0, 1) = (0, 2)$, then the matrix of T is $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

(b) First, $T(1, 0, -1) = (1, -3) = -3 \cdot (0, 1) + 1 \cdot (1, 0)$. Next, $T(1, 1, 1) = (2, 1)$ $= 1 \cdot (0, 1) + 2 \cdot (1, 0)$. Lastly, $T(1, 0, 0) = (1, -1) = -1(0, 1) + 1(1, 0)$. So, $[T]_{B \to C}$ is $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

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3. Determine whether the statement is true or false, and justify your answer.

   (a) The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

   (b) Two square matrices that have the same determinant must have the same size.

   (c) For every square matrix A and every scalar c, $\det(cA) = c\det(A)$.

   (d) For all square matrices A and B, $\det(A + B) = \det(A) + \det(B)$.

   (e) For every 2×2 matrix A, $\det(A^2) = (\det(A))^2$.

Solution:

(a) False. The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

(b) False. The determinants of the 2×2 identity matrix and the 3×3 identity matrix are both 1.

(c) False. Let I be the 2×2 identity matrix. Then $\det(2I) = 4$, but $2\det(I) = 2$.

(d) False. Let I be the 2×2 identity matrix. Then $\det(I + I) = \det(2I) = 4$, but $\det(I) + \det(I) = 2$.

(e) True. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$, and

$$\det(A^2) = a^2d^2 - 2abcd + b^2c^2 = \det(A)^2.$$

4. Consider the matrix $A = \begin{bmatrix} t & t & 0 \\ t^2 & 3 & 2t \\ 0 & t & t \end{bmatrix}$.

   (a) Find the determinant of A.

   (b) Find all values of t for which A is invertible. (Recall that a square matrix A is invertible if and only if $\det(A) \neq 0$.)

Solution:

(a) Computing a cofactor expansion along the first row yields:

$$\begin{aligned} \det(A) &= t \begin{vmatrix} 3 & 2t \\ t & t \end{vmatrix} - t \begin{vmatrix} t^2 & 2t \\ 0 & t \end{vmatrix} \\ &= t(3t - 2t^2) - t(t^3) \\ &= -t^4 - 2t^3 + 3t^2. \end{aligned}$$

(b) We need to find the roots of $-t^4 - 2t^3 + 3t^2$. Factoring, we have

$$-t^4 - 2t^3 + 3t^2 = -t^2(t^2 + 2t - 3) = -t^2(t-1)(t+3).$$

The roots are $0, 1, -3$. So, A is invertible when $t \neq 0, 1, -3$.

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5. Use Cramer's rule to find a solution to the following system of equations:

$$\begin{aligned} x - y + 2z &= -3 \\ x + 2y + 3z &= 4 \\ 2x + y + z &= -3 \end{aligned}$$

Solution: We can write the above equation in matrix form as follows:
$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$.

Replacing the first column with the vector $\begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$, we obtain: $\begin{bmatrix} -3 & -1 & 2 \\ 4 & 2 & 3 \\ -3 & 1 & 1 \end{bmatrix}$. This matrix has determinant 36. The original matrix has determinant -12. So $x = \frac{36}{-12} = -3$.

Replacing the second column with the vector $\begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$, we obtain: $\begin{bmatrix} 1 & -3 & 2 \\ 1 & 4 & 3 \\ 2 & -3 & 1 \end{bmatrix}$. This matrix has determinant -24. The original matrix has determinant -12. So $y = \frac{-24}{-12} = 2$.

Replacing the third column with the vector $\begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$, we obtain: $\begin{bmatrix} 1 & -1 & -3 \\ 1 & 2 & 4 \\ 2 & 1 & -3 \end{bmatrix}$. This matrix has determinant -12. The original matrix has determinant -12. So $z = \frac{-12}{-12} = 1$.

Finally, we can check that $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ is a solution: $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$.