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M20580 L.A. and D.E. Tutorial
Worksheet 9

1. (a) Give an example of a matrix with only one real eigenvalue whose algebraic multiplicity is equal to its geometric multiplicity. Show how you compute the two multiplicities.
   (b) Give an example of a matrix with only one real eigenvalue whose algebraic multiplicity is greater than its geometric multiplicity. Show how you compute the two multiplicities.
   (c) Give an example of a matrix with no real eigenvalue and compute its complex eigenvalue.

   Hint: some simple 2×2 matrices should do the job.

Solution:

(a) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has only one eigen-value and the corresponding eigen-space has a basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence the geometric multiplicity is 2.

Its characteristic polynomial is $\det(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}) = (\lambda - 1)^2$. Hence the algebraic multiplicity is 2.

(b) The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigen-value, and the corresponding eigen-space is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence the geometric multiplicity is 1.

Its characteristic polynomial is $\det(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}) = (\lambda - 1)^2$. Hence the algebraic multiplicity is 2.

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has two complex eigenvalues: i and -i.

Note that this matrix corresponds to a counterclockwise rotation by $90°$; complex eigenvalues typically arise when there is some kind of rotation.

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2. Determine whether the statements are true or false, and give a counterexample if false.

   (a) Every linearly independent set of vectors is orthogonal.

   (b) Every orthogonal set of vectors is linearly independent.

   (c) Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

   (d) $\text{proj}_W \mathbf{x}$ is orthogonal to every vector in W.

   (e) Every orthogonal set is orthonormal.

Solution:

(a) False. In $\mathbb{R}^2$, the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ form a basis, hence they are linearly independent, but they are not orthogonal since $\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 1 \neq 0$.

(b) False. Any orthogonal set of **nonzero** vectors is linearly independent. The set $\{0\}$ is orthogonal but not linearly independent.

(c) True. Any nontrivial subspace has a basis, and we can use the Gram-Schmidt algorithm to find an orthonormal basis.

(d) False. W contains the vector $\text{proj}_W \mathbf{x}$. What is true is that $\mathbf{x} - \text{proj}_W \mathbf{x}$ is orthogonal to every vector in W. Take $W = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix}\}$ and $\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}$; then $\text{proj}_W \mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ is not orthogonal to W.

(e) False. $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\}$ is orthogonal but not orthonormal.

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3. Let $V = \text{span}\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\\2\end{bmatrix}\right\}$ be a subspace of $\mathbb{R}^4$. Find a basis of the orthogonal complement $V^\perp$.

Solution: If $\mathbf{x}$ is in $V^\perp$, then $\mathbf{x}\cdot\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = 0$ and $\mathbf{x}\cdot\begin{bmatrix}2\\3\\-1\\2\end{bmatrix} = 0$. This implies that $\mathbf{x}$ is in the nullspace of $\begin{bmatrix}1 & 0 & -1 & 1\\2 & 3 & -1 & 2\end{bmatrix}$. The augmented matrix $\left[\begin{array}{cccc|c}1 & 0 & -1 & 1 & 0\\2 & 3 & -1 & 2 & 0\end{array}\right]$ row reduces to $\left[\begin{array}{cccc|c}1 & 0 & -1 & 1 & 0\\0 & 1 & 1/3 & 0 & 0\end{array}\right]$. From this, we see that:

$$x = \begin{bmatrix}s-t\\-s/3\\s\\t\end{bmatrix} = \begin{bmatrix}s\\-s/3\\s\\0\end{bmatrix} + \begin{bmatrix}-t\\0\\0\\t\end{bmatrix} = s\begin{bmatrix}1\\-1/3\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

Therefore, $V^\perp = \text{span}\left\{\begin{bmatrix}1\\-1/3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$

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4. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix}$ by using normal equations.

Solution: We have $A^T = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix}$. So $A^TA = \begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix}$, and $A^T\mathbf{b} = \begin{bmatrix} 12 \\ -24 \end{bmatrix}$. Then the normal equation is

$$\begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 12 \\ -24 \end{bmatrix}.$$

Thus

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 19/234 & -2/117 \\ -2/117 & 7/234 \end{bmatrix} \begin{bmatrix} 12 \\ -24 \end{bmatrix} = \begin{bmatrix} 18/13 \\ -12/13 \end{bmatrix}$$

is a least squares solution.