Math 20580 Name:________________________
Final Exam Instructor:________________________
December 13, 2023 Section:________________________

Calculators are NOT allowed. You will be allowed 2 hours to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1.	a	b	c	d	e	11.	a	b	c	d	e
2.	a	b	c	d	e	12.	a	b	c	d	e
3.	a	b	c	d	e	13.	a	b	c	d	e
4.	a	b	c	d	e	14.	a	b	c	d	e
5.	a	b	c	d	e	15.	a	b	c	d	e
6.	a	b	c	d	e	16.	a	b	c	d	e
7.	a	b	c	d	e	17.	a	b	c	d	e
8.	a	b	c	d	e	18.	a	b	c	d	e
9.	a	b	c	d	e	19.	a	b	c	d	e
10.	a	b	c	d	e	20.	a	b	c	d	e

1. Let W be the column space of a 23×2023 matrix of rank 3. What is the dimension of $W^{\perp}$ (the orthogonal complement of W)?

(a) 3 (b) 20 (c) 23 (d) 2020 (e) 2023

2. Let M be the following matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of the following are eigenvalues of M?

I. 0 II. 1 III. 2 IV. 3

(a) I, II, and IV only (b) I, II, and III only (c) II, III, and IV only

(d) I and IV only (e) II and IV only

3. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

   Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$, the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

   (a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$

4. Describe the implicit solution of the exact equation

$$(e^x \sin(y) + 2y)dx + (2x + e^x \cos(y))dy = 0.$$

   (a) $e^x \sin(y) = x^2 + y^2 + C$ (b) $e^x \sin(y) + 2xy + g(y)$ (c) $2xy + e^x \sin(y) = C$

   (d) $y = \dfrac{e^x + x^2}{\sin(y)} + C$ (e) $e^x \tan(y) = C$

5. Describe the largest interval where a solution for the following initial value problem is guaranteed to exist:

$$\begin{cases} (\cos x)y'' + y' + (\ln|x|)y = x^2 \\ y(2) = 1, \ y'(2) = -1 \end{cases}$$

(a) $(0, \pi)$ (b) $(-\infty, \infty)$ (c) $(-\pi/2, \pi/2)$ (d) $(0, \infty)$ (e) $(\pi/2, 3\pi/2)$

6. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2, and consider its basis $\mathcal{B} = \{1, 2-t, (2-t)^2\}$. The coordinate vector of $3t^2 - 8t + 6$ relative to the basis $\mathcal{B}$ is:

(a) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ -8 \\ 6 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

7. Consider the solution $y(x)$ of the autonomous equation

$$\frac{dy}{dx} = y^2 \cdot (y^2 - 2)$$

satisfying the initial condition $y(1) = -1$. Compute $\lim_{x\to\infty} y(x)$.

(a) $\sqrt{2}$ (b) $-\infty$ (c) $-\sqrt{2}$ (d) 0 (e) ∞

8. Given the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, determine its nullity and rank.

(a) $\text{nullity}(A) = 1, \text{rank}(A) = 3$ (b) $\text{nullity}(A) = 2, \text{rank}(A) = 3$

(c) $\text{nullity}(A) = 0, \text{rank}(A) = 5$ (d) $\text{nullity}(A) = 1, \text{rank}(A) = 4$

(e) $\text{nullity}(A) = 2, \text{rank}(A) = 2$

9. The matrix $A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & -3 & -1 & 9 \end{bmatrix}$ has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

For which value of the parameter t is the vector $[1 \quad -1 \quad t \quad 0]$ in the row space of A?

(a) 2 (b) $\frac{7}{2}$ (c) 0 (d) -7 (e) -2

10. If $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then b_{21} is:

(a) 0 (b) $-1/2$ (c) -2 (d) $3/2$ (e) 2

11. Find the solution of the initial value problem

$$\begin{cases} y'' - 6y' + 9y = 0, \\ y(0) = 0,\ \ y'(0) = 2. \end{cases}$$

(a) $2e^{-3t} - 2$ (b) $e^{3t} - e^{-3t}$ (c) $2te^{3t}$ (d) $-6e^{3t} + 3$ (e) $-2e^{t} + e^{-2t}$

12. Find the Wronskian $W(f_1, f_2, f_3)$ where $f_1(x) = x,\ f_2(x) = x^2$ and $f_3(x) = 1/x$.

(a) x^2 (b) 0 (c) $\dfrac{x^2 + x^3 + 1}{x}$ (d) $\dfrac{x^2}{2} + \dfrac{x^3}{3} + \ln(x)$ (e) $\dfrac{6}{x}$

13. Consider the differential equation $y'' + 4y = 4\sin(2x)$. Use the method of undetermined coefficients to find a particular solution.

(a) Ae^{4x} (b) $-x\cos(2x)$ (c) $e^{2x}\sin(2x)$ (d) $-2\cos(2x)$

(e) $x\sin(2x) + \cos(2x)$

14. Find the solution of the initial value problem

$$\begin{cases} y' = \dfrac{2x}{y + x^2 y} \\ y(0) = 1 \end{cases}$$

(a) $x^2\ln(x) + 1$ (b) $\sqrt{2\ln(x^2+1)+1}$ (c) $(x-1)^2$ (d) $\ln(x^2+1)+1$

(e) there is no solution

15. Which of the following matrices is similar to $A = \begin{bmatrix} -3 & 41 \\ -1 & 7 \end{bmatrix}$.

(a) $\begin{bmatrix} -1 & -3 \\ 7 & 41 \end{bmatrix}$ (b) $\begin{bmatrix} 20 & 21 \\ -4 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}$

16. Find the matrix R in the QR decomposition of $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, provided that

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(a) $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{8} \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(d) $\begin{bmatrix} \sqrt{2} & -\sqrt{3} & -1 \\ 0 & \sqrt{3} & \sqrt{2} \\ 0 & 0 & \sqrt{8} \end{bmatrix}$ (e) none of the above

17. Which of the following describes the least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ only

(b) $\begin{bmatrix} 1/2 \\ -3 \end{bmatrix}$ only

(c) $\begin{bmatrix} 5/6 \\ 1/3 \end{bmatrix}$ only

(d) infinitely many solutions

(e) no solutions

18. Which formula describes the general solution of the differential equation

$$ty'' - (1+t)y' + y = 0, \ t > 0$$

given the fact that $y_1(t) = e^t$ is a solution of this equation.

(a) $c_1 + c_2 e^t$

(b) $c_1 e^t + c_2 t e^t$

(c) $c_1 e^t + c_2 \ln(t+1)$

(d) $c_1 e^t + c_2 e^{-t}$

(e) $c_1 e^t + c_2 (t+1)$

19. Consider the differential equation $x^2y'' - 2y = 3x^2 - 1$. The functions

$$y_1 = x^2 \quad \text{and} \quad y_2 = x^{-1}$$

form a fundamental set of solutions for the associated homogeneous equation. Which of the following describes a particular solution of the non-homogeneous equation?

(a) $x^2 - \dfrac{1}{x}$ (b) $c_1x^2 + c_2x^{-1}$ (c) $x^2\ln(x) + \dfrac{1}{2}$ (d) $\dfrac{x^2}{3x^2 - 1}$

(e) $\dfrac{x^3}{3} + \ln(x)$

20. Find the general solution of the equation

$$(4 + x^2)\frac{dy}{dx} + 2xy = 4x.$$

(a) $\dfrac{2x^2 + C}{4 + x^2}$ (b) $\ln(4 + x^2) + C$ (c) $\dfrac{C}{4 + x^2}$ (d) $\dfrac{2x}{4 + x^2} + C$

(e) cannot be found explicitly using methods we learned