Math 20580 Name:____________________
Midterm 1 Instructor:____________________
September 19, 2023 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{cases} 2x + 3y = a \\ 4x + 5y = b \end{cases}$$

If (x, y) is a solution, which of the following describes y in terms of a, b?

(a) $y = (-5a + 3b)/2$ (b) $y = 6a + 8b$ (c) $y = 2a - b$ (d) $y = 2a + 5b$

(e) y is not determined by a, b

2. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Which of the following sets of vectors span $\mathbb{R}^2$?

(I) $\{\vec{v}_1, \vec{v}_2\}$ (II) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (III) $\{\vec{v}_2\}$ (IV) $\{\vec{v}_2, \vec{v}_3\}$

(a) III only (b) I and III only (c) I and II only (d) II and IV only

(e) III and IV only

3. Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map given by counterclockwise rotation of the plane about the origin by an angle of $\dfrac{\pi}{4}$ (in radians). Let A be the standard matrix of T. Which of the following matrices is equal to A^2?

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4. Under which of the scenarios below is the equation $A\vec{x} = \vec{b}$ guaranteed to have at least one solution?

I. A is a 3×4 matrix and $\vec{b}$ is any vector in the column space of A.

II. A is a 4×3 matrix of rank 2 and $\vec{b}$ is any vector in $\mathbb{R}^4$.

III. A is a 3×3 matrix and $\vec{b}$ is a vector in the null space of A.

IV. A is an invertible 2×2 matrix and $\vec{b}$ is the zero vector in $\mathbb{R}^2$.

(a) I and IV only (b) IV only (c) I, III, and IV only

(d) II and IV only (e) III only

5. Given the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

determine $A^{-1}B - AB^T$.

(a) $\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 4 & 2 \\ -6 & 0 \end{bmatrix}$

6. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1-t \\ 1+t \\ 1 \end{bmatrix}$$

For which value of t does the vector $\vec{v}_3$ belong to $\text{Span}(\vec{v}_1, \vec{v}_2)$?

(a) all $t \leq 2$ (b) $t = 2$ and $t = -1$ (c) $t = -3$ only (d) $t = 0$ only

(e) no value of t

7. Which of the following sets is linearly independent?

(I) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ (II) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ (III) $\left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ (IV) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

(a) I, II, III only (b) II and IV only (c) I and II only

(d) III and IV only (e) I, III, IV only

8. Let A be a 7×8 matrix of rank 3. Which of the following is equal to the dimension of the null space of the transpose matrix A^T?

(a) 0 (b) 3 (c) 4 (d) 5 (e) 7

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

(a) Find a basis for $\mathrm{Col}(A)$ (the column space of A).

(b) Find a basis for $\mathrm{Row}(A)$ (the row space of A).

(c) Find a basis for $\mathrm{Nul}(A)$ (the null space of A).

10. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

11. Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix},$$

and the matrix transformation $S(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix}.$$

(a) Find the standard matrix of T.

(b) Find the standard matrices of the compositions $S \circ T$ and $T \circ S$.

(c) Find a vector $\vec{v}$ in $\mathbb{R}^3$ with $T(\vec{v}) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

12. Consider the bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ (recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{x}]_{\mathcal{B}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

(b) If $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, determine the coordinate vectors $[\vec{v}]_{\mathcal{B}}$ and $[\vec{v}]_{\mathcal{C}}$.