**Math 20580** Name:________________________

**Midterm 3** Instructor:________________________

**November 14, 2023** Section:________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

**Sign the pledge.** "On my honor, I have neither given nor received unauthorized aid on this Exam":

________________________________________

1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

## Part I: Multiple choice questions (7 points each)

1. Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 3/7 \\ 1/7 \\ -2/7 \end{bmatrix}$ (b) $\vec{0}$ (c) $\begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

2. If $A$ is a $4 \times 6$ matrix of rank 2, what is the dimension of the orthogonal complement of the row space of $A^T$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

3. Find the distance between $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$ (e) $\sqrt{6}$

4. Which of the following functions is the general solution of the equation $y' + 5y = 0$?

(a) $e^{5x} + C$ (b) $C \cdot e^{-5x}$ (c) $5C + e^{-x}$ (d) $e^{-5x} - C$ (e) $e^{x} - 5$

5. Find the solution of the initial value problem

$$\begin{cases} y' + 2y = 3e^x, \\ y(1) = 0 \end{cases}$$

(a) $(x-1)e^x$ (b) $e^x$ (c) $x^2$ (d) $e^x - e^{3-2x}$ (e) $(3e^x - 3)/2$

6. Consider the initial value problem

$$\frac{dy}{dt} = 2y^2 - 4y, \qquad y(5) = 1.$$

Which of the following describes the nature of the solution?

(a) $\lim_{t\to-\infty} y(t) = 2;$ $\lim_{t\to\infty} y(t) = 0$

(b) $\lim_{t\to-\infty} y(t) = 2;$ $\lim_{t\to\infty} y(t) = \infty$

(c) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = 4$

(d) $\lim_{t\to-\infty} y(t) = -\infty;$ $\lim_{t\to\infty} y(t) = 0$

(e) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = -\infty$

7. The differential equation

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = y + x + t$$

is

(a) an equation of order 2 (b) a partial differential equation (c) exact
(d) separable (e) an ordinary differential equation

8. Consider the orthogonal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Find the coordinate vector $[\vec{w}]_{\mathcal{B}}$ if $\vec{w} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

**Part II: Partial credit questions (11 points each). Show your work.**

9. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for $W$.

(b) Find the $QR$ decomposition of the matrix $A$ with columns $\vec{w}_1, \vec{w}_2, \vec{w}_3$.

10. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

(b) Find the vector in the column space of $A$ which is closest to $\vec{b}$.

11. Consider the differential equation

$$(3x\sin(y) + 2e^y)\,dx + \left(x^2\cos(y) + xe^y\right)dy = 0.$$

(a) Explain why the equation is not exact.

(b) Find an integrating factor $\mu$ which only depends on the variable $x$.

(c) Determine the implicit solution which satisfies the initial condition $y(1) = 0$.

12. (a) Find the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} + 2xy^2 = 0 \\ y(0) = -1 \end{cases}$$

(b) Find the maximal interval on which the solution in (a) is defined.