Math 20580
Midterm 1
September 19, 2023

Name:______________________________
Instructor:______________________________
Section:______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. ☐ a ☐ b ☒ c ☐ d ☐ e
2. ☐ a ☐ b ☐ c ☒ d ☐ e
3. ☒ a ☐ b ☐ c ☐ d ☐ e
4. ☒ a ☐ b ☐ c ☐ d ☐ e
5. ☐ a ☐ b ☐ c ☐ d ☒ e
6. ☐ a ☐ b ☐ c ☒ d ☐ e
7. ☐ a ☒ b ☐ c ☐ d ☐ e
8. ☐ a ☐ b ☒ c ☐ d ☐ e

____

Multiple Choice.

9.

10.

11.

12.

____

Total.

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{cases} 2x + 3y = a \\ 4x + 5y = b \end{cases}$$

If (x, y) is a solution, which of the following describes y in terms of a, b?

(a) $y = (-5a + 3b)/2$ (b) $y = 6a + 8b$ [✓ (c) $y = 2a - b$] (d) $y = 2a + 5b$

(e) y is not determined by a, b

$$\left[\begin{array}{cc|c} 2 & 3 & a \\ 4 & 5 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 3 & a \\ 0 & -1 & b-2a \end{array}\right] \longrightarrow -y = b - 2a$$

So

$$\boxed{y = 2a - b}$$

2. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Which of the following sets of vectors span $\mathbb{R}^2$?

~~(I)~~ $\{\vec{v}_1, \vec{v}_2\}$ (II) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ ✓ ~~(III)~~ $\{\vec{v}_2\}$ (IV) $\{\vec{v}_2, \vec{v}_3\}$ ✓

(a) III only (b) I and III only (c) I and II only [✓ (d) II and IV only]

(e) III and IV only

$\vec{v_2} = 2\vec{v_1}$ so Span$\{\vec{v_1}, \vec{v_2}\}$ = Span$\{\vec{v_2}\}$ is a line in $\mathbb{R}^2$

So (I), $\boxed{III}$ are wrong!

$\{\vec{v_2}, \vec{v_3}\}$ independent (not scalar multiples of one another)

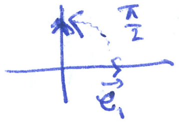
So they form basis for $\mathbb{R}^2$, hence they span $\mathbb{R}^2$

Same is then true about $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ (i.e. they also span)

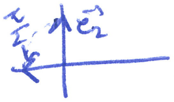
3. Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map given by counterclockwise rotation of the plane about the origin by an angle of $\frac{\pi}{4}$ (in radians). Let A be the standard matrix of T. Which of the following matrices is equal to A^2?

[✓ (a)] $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

A^2 is the standard matrix for rotation by $2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$



$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



4. Under which of the scenarios below is the equation $A\vec{x} = \vec{b}$ guaranteed to have at least one solution?

I. A is a 3×4 matrix and $\vec{b}$ is any vector in the column space of A. ✓

II. A is a 4×3 matrix of rank 2 and $\vec{b}$ is any vector in $\mathbb{R}^4$. ✗

III. A is a 3×3 matrix and $\vec{b}$ is a vector in the null space of A. ✗

IV. A is an invertible 2×2 matrix and $\vec{b}$ is the zero vector in $\mathbb{R}^2$. ✓ $\vec{x} = \vec{0}$ is a solution

[✓ (a)] I and IV only (b) IV only (c) I, III, and IV only

(d) II and IV only (e) III only

II is false : Col A is a subspace of $\mathbb{R}^4$ of dimension 2
if $\vec{b}$ is not in Col A then $A\vec{x} = \vec{b}$ inconsistent

III is false : take $A = 0$ the zero matrix
$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: $A\vec{x} = \vec{b}$ inconsistent

5. Given the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

determine $A^{-1}B - AB^T$.

(a) $\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ [✓ (e) $\begin{bmatrix} 4 & 2 \\ -6 & 0 \end{bmatrix}$]

$$A^{-1} = \frac{1}{1}\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{so } A^{-1}B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -1 & +1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$$

$$A^{-1}B - AB^T = \begin{bmatrix} 4 & 2 \\ -6 & 0 \end{bmatrix}$$

6. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1-t \\ 1+t \\ 1 \end{bmatrix}$$

For which value of t does the vector $\vec{v}_3$ belong to Span$(\vec{v}_1, \vec{v}_2)$?

(a) all $t \leq 2$ (b) $t = 2$ and $t = -1$ (c) $t = -3$ only [✓ (d) $t = 0$ only]

(e) no value of t

$$\left[\begin{array}{cc|c} \textcircled{1} & 1 & 1-t \\ -1 & 1 & 1+t \\ 2 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 1-t \\ 0 & 2 & 2 \\ 0 & -1 & 2t-1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{cc|c} \textcircled{1} & 1 & 1-t \\ 0 & \textcircled{1} & 1 \\ 0 & -1 & 2t-1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} \textcircled{1} & 1 & 1-t \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 2t \end{array}\right]$$

↑ don't want pivot in last column

$2t = 0$ ←

so $\boxed{t=0}$

contains $\vec{0}$

7. Which of the following sets is linearly independent?

✗ (I) $\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix} \right\}$ $= 3\cdot\begin{bmatrix}1\\2\end{bmatrix}$ (II) $\left\{ \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$ ✓ ✗ (III) $\left\{ \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\}$ ✓ (IV) $\left\{ \begin{bmatrix}1\\2\end{bmatrix} \right\}$ ✓

(a) I, II, III only [✓ (b)] II and IV only (c) I and II only

(d) III and IV only (e) I, III, IV only

8. Let A be a 7×8 matrix of rank 3. Which of the following is equal to the dimension of the null space of the transpose matrix A^T?

(a) 0 (b) 3 [✓ (c)] 4 (d) 5 (e) 7

A^T is 8×7 (has 7 columns)

$\text{rank}(A^T) = \text{rank}(A) = 3$

by rank theorem

$\dim(\text{Null}\, A^T) = 7 - 3 = 4$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$ move here

(a) Find a basis for Col(A) (the column space of A).

$$\begin{bmatrix} \circled{1} & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_4 \to R_4 - R_1} \begin{array}{ccccc} & & & \text{free variables} & \\ & & & \downarrow & \downarrow \\ \left[\begin{matrix} \boxed{1} \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ \boxed{1} \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ \boxed{1} \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 2 \\ -1 \\ 2 \\ 0 \end{matrix}\right] \\ \uparrow & \uparrow & \uparrow & & \\ & \text{pivot positions} & & & \end{array} \quad \text{REF}$$

basis for Col A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) Find a basis for Row(A) (the row space of A).

non zero rows in REF : $\left\{ [1\ 0\ 1\ 0\ 2], [0\ 1\ 0\ 1\ -1], [0\ 0\ 1\ 1\ 2] \right\}$

(c) Find a basis for Nul(A) (the null space of A). Solve $A\vec{x} = \vec{0}$ using REF

$x_4 = s$ free

$x_5 = t$ free

$x_3 + x_4 + 2x_5 = 0 \Rightarrow x_3 = -s - 2t$

$x_2 + x_4 - x_5 = 0 \Rightarrow x_2 = t - s$

$x_1 + x_3 + 2x_5 = 0 \Rightarrow x_1 = s$

$$\text{Nul}A = \left\{ \begin{bmatrix} s \\ t-s \\ -s-2t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ so basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

10. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1}$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 1 & -1 & 1 & 0 \\ 0 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{5}{2} R_2}$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 1 & -1 & 1 & 0 \\ 0 & 0 & \textcircled{$-\frac{1}{2}$} & 5/2 & -5/2 & 1 \end{array}\right] \xrightarrow{R_3 \to -2R_3}$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 1 & -1 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -5 & 5 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 4 & -4 & 2 \\ 0 & 0 & 1 & -5 & 5 & -2 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2} R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -5 & 5 & -2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 2 & -2 & 1 \\ -5 & 5 & -2 \end{matrix}}_{\text{inverse}}$$

11. Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix},$$

and the matrix transformation $S(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix}.$$

(a) Find the standard matrix of T.

$$\begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Find the standard matrices of the compositions $S \circ T$ and $T \circ S$.

$$[S\circ T] = [S]\cdot[T] = \begin{bmatrix} -1 & -3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & 5 & 3 \\ -2 & -4 & -2 \end{bmatrix}$$

$$[T\circ S] = [T]\cdot[S] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\left(\text{So } T(S(\vec{x})) = \vec{x} \text{ for all } \vec{x}\right)$

(c) Find a vector $\vec{v}$ in $\mathbb{R}^3$ with $T(\vec{v}) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

can take $\vec{v} = S\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 12 \\ -9 \\ 4 \end{bmatrix}$

$\left(\text{or } \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ any solution of } \begin{cases} x_1 + x_2 = 3 \\ x_2 + x_3 = -5 \end{cases}\right)$

12. Consider the bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\5\\2\end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ (recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{x}]_{\mathcal{B}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 5 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{array}\right]$$

move last row to the top

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 5 \end{array}\right] \xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & -5 \end{array}\right]$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$

(b) If $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, determine the coordinate vectors $[\vec{v}]_{\mathcal{B}}$ and $[\vec{v}]_{\mathcal{C}}$.

to find $[\vec{v}]_{\mathcal{B}}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 5 & 2 \\ 1 & 0 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 5 & 0 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & -2 & 5 & 0 \end{array}\right] \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right) \rightarrow \boxed{c_3 = -4}$$

$\boxed{c_1 = 11}$

$\boxed{c_2 = -10}$

$$\text{So } \boxed{[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}11\\-10\\-4\end{bmatrix}}$$

$$\boxed{[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{v}]_{\mathcal{B}} = \begin{bmatrix}1 & 0 & 2\\1 & 1 & 0\\-2 & 0 & -5\end{bmatrix}\begin{bmatrix}11\\-10\\-4\end{bmatrix} = \begin{bmatrix}3\\1\\-2\end{bmatrix}}$$