Math 20580 Name:____________________
Midterm 3 Instructor:____________________
November 14, 2023 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

---

1. a b ~~c~~ d e
2. a ~~b~~ c d e
3. ~~a~~ b c d e
4. a ~~b~~ c d e
5. a b c ~~d~~ e
6. ~~a~~ b c d e
7. a ~~b~~ c d e
8. a b c d ~~e~~

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 3/7 \\ 1/7 \\ -2/7 \end{bmatrix}$ (b) $\vec{0}$ (c) $\begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

$$\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{2}{3}\vec{u} = \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

2. If A is a 4×6 matrix of rank 2, what is the dimension of the orthogonal complement of the row space of A^T?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

$$\text{Row}(A^T)^\perp = \text{nul}(A^T)$$

$$\left.\begin{array}{l} A^T \text{ has 4 columns} \\ \text{rank}(A^T) = 2 \end{array}\right\} \Rightarrow \text{nullity}(A^T) = 4-2 = 2$$

orthogonal basis

3. Find the distance between $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and the subspace $W = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \right\}$.

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$ (e) $\sqrt{6}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_1$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_2$

$$\text{proj}_W(\vec{w}) = \frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{3}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{proj}_{W^\perp}(\vec{w}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \text{ and } \left\|\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right\| = 1$$

4. Which of the following functions is the general solution of the equation $y' + 5y = 0$?

(a) $e^{5x} + C$ (b) $C \cdot e^{-5x}$ (c) $5C + e^{-x}$ (d) $e^{-5x} - C$ (e) $e^x - 5$

$$y' = -5y$$

$$\Rightarrow y = C \cdot e^{-5x}$$

5. Find the solution of the initial value problem

$$\begin{cases} y' + 2y = 3e^x, \\ y(1) = 0 \end{cases}$$

(a) $(x-1)e^x$ (b) e^x (c) x^2 (d) $e^x - e^{3-2x}$ (e) $(3e^x - 3)/2$

$\mu(x) = e^{2x}$

$$y = \frac{\int e^{2x} \cdot 3e^x \, dx}{e^{2x}} = \frac{e^{3x} + C}{e^{2x}}$$

$y(1) = 0 \implies C = -e^3$

So $y = e^x - e^{3-2x}$

6. Consider the initial value problem

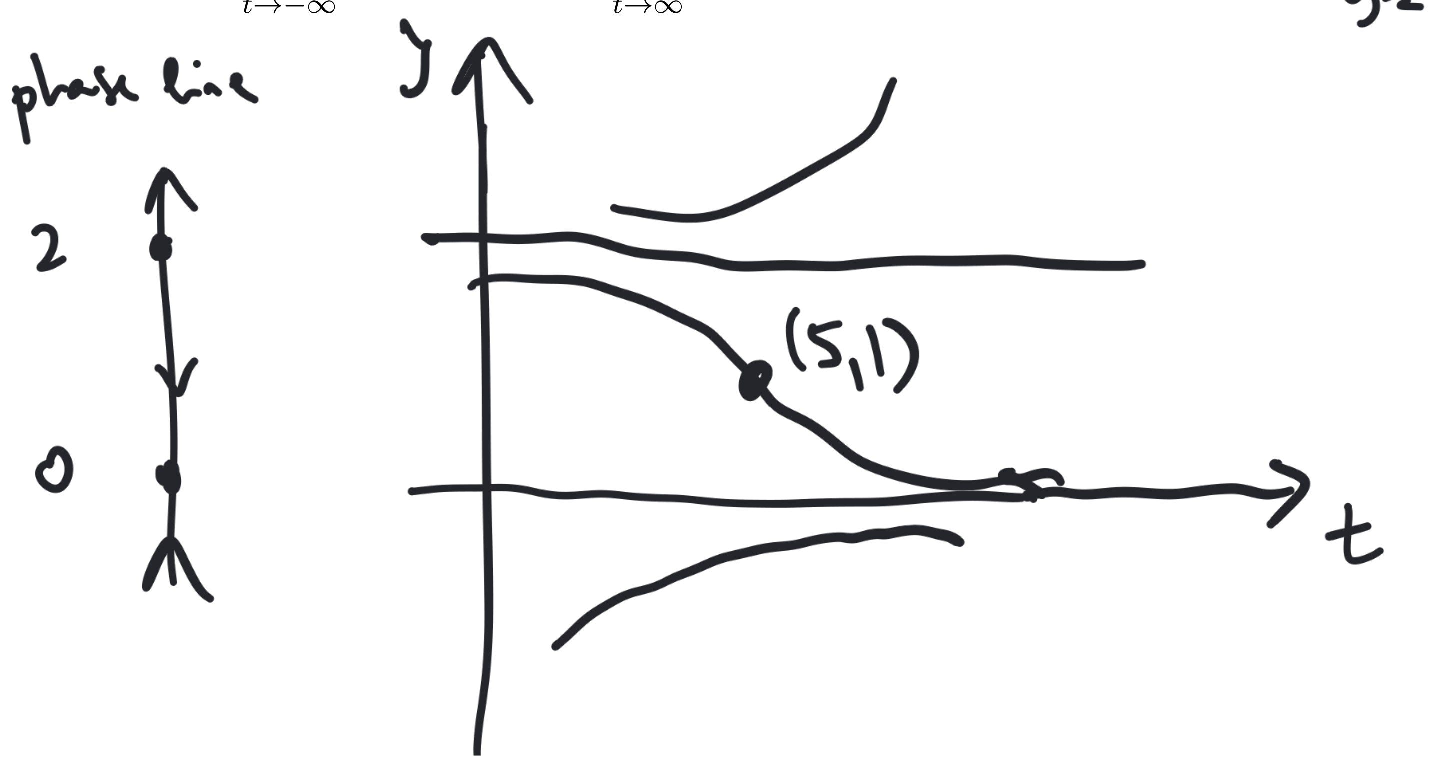
$$\frac{dy}{dt} = 2y^2 - 4y, \qquad y(5) = 1.$$

Which of the following describes the nature of the solution?

(a) $\lim_{t\to-\infty} y(t) = 2;$ $\lim_{t\to\infty} y(t) = 0$
(b) $\lim_{t\to-\infty} y(t) = 2;$ $\lim_{t\to\infty} y(t) = \infty$
(c) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = 4$
(d) $\lim_{t\to-\infty} y(t) = -\infty;$ $\lim_{t\to\infty} y(t) = 0$
(e) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = -\infty$

$2y^2 - 4y = 2y(y-2)$

critical values $y = 0, 2$



partial derivatives

x,t independent variables

7. The differential equation

$$\frac{\partial y}{\partial t}+\frac{\partial y}{\partial x}=y+x+t$$

is

(a) an equation of order 2 (b) a partial differential equation (c) exact
(d) separable (e) an ordinary differential equation

8. Consider the orthogonal basis $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ of $\mathbb{R}^3$, where

$$\vec{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \vec{v}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix},\quad \vec{v}_3=\begin{bmatrix}1\\1\\-2\end{bmatrix}.$$

Find the coordinate vector $[\vec{w}]_{\mathcal{B}}$ if $\vec{w}=\begin{bmatrix}0\\-2\\5\end{bmatrix}$.

(a) $\begin{bmatrix}1\\1\\1\end{bmatrix}$ (b) $\begin{bmatrix}3\\7\\-10\end{bmatrix}$ (c) $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ (d) $\begin{bmatrix}1\\2\\3\end{bmatrix}$ (e) $\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$$\vec{w}=\frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1+\frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2+\frac{\vec{w}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3$$

$$=\frac{3}{3}\vec{v}_1+\frac{2}{2}\vec{v}_2+\frac{-12}{6}\vec{v}_3=\vec{v}_1+\vec{v}_2-2\vec{v}_3$$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for W. $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

orthogonal $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\quad \vec{v}_1 = \vec{w}_1 \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

(b) Find the QR decomposition of the matrix A with columns $\vec{w}_1, \vec{w}_2, \vec{w}_3$.

From (a), $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$

$$R = Q^TA = \begin{bmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 25/21 \\ -10/21 \end{bmatrix}$$

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

$$\text{proj}_{\text{col}A}(\vec{b}) = A\hat{x}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 25/21 \\ -10/21 \end{bmatrix}$$

$$= \begin{bmatrix} 25/21 \\ 40/21 \\ -20/21 \end{bmatrix}$$

11. Consider the differential equation

$$\underbrace{(3x\sin(y) + 2e^y)}_{M}\, dx + \underbrace{\left(x^2\cos(y) + xe^y\right)}_{N}\, dy = 0.$$

(a) Explain why the equation is not exact.

$M_y = 3x\cos(y) + 2e^y \neq N_x = 2x\cos y + e^y$

(b) Find an integrating factor μ which only depends on the variable x.

$\mu' = \mu \cdot \frac{M_y - N_x}{N} = \mu \cdot \frac{x\cos y + e^y}{x^2\cos y + xe^y} = \mu \cdot \frac{1}{x}$

So $\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

(c) Determine the implicit solution which satisfies the initial condition $y(1) = 0$.

new exact equation: $\underbrace{(3x^2\sin y + 2xe^y)}_{M}\,dx + \underbrace{(x^3\cos y + x^2e^y)}_{N}\,dy = 0$

$f(x,y) = \int (3x^2\sin y + 2xe^y)\,dx = x^3\sin y + x^2e^y + g(y)$

$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{x^3\cos y} + \cancel{x^2e^y} + g'(y) = \cancel{x^3\cos y} + \cancel{x^2e^y}$

$\Rightarrow g'(y) = 0$, can take $g(y) = 0$

Implicit soln: $\left.\begin{array}{l} x^3\sin y + x^2e^y = C \\ x=1, y=0 \end{array}\right\} \Rightarrow 0 + 1 = C$ so $\boxed{x^3\sin y + x^2e^y = 1}$

12. (a) Find the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} + 2xy^2 = 0 \\ y(0) = -1 \end{cases}$$

$$\frac{dy}{dx} = -2xy^2 \Rightarrow \int \frac{dy}{-y^2} = \int 2x\,dx$$

$$\Rightarrow \frac{1}{y} = x^2 + C$$

$$x=0,\ y=-1 \quad \Big| \Rightarrow C = -1$$

$$\Downarrow$$

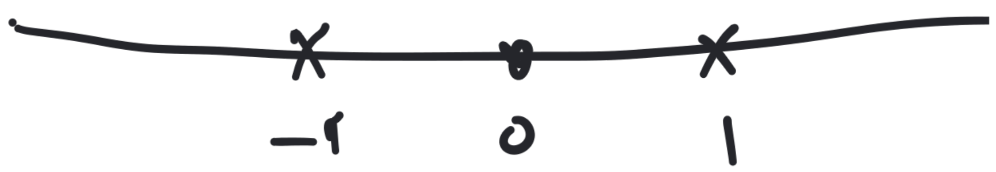
$$\frac{1}{y} = x^2 - 1$$

$$\Rightarrow \boxed{y = \frac{1}{x^2-1}}$$

(b) Find the maximal interval on which the solution in (a) is defined.

Want interval I containing $x_0 = 0$

and avoiding $x = \pm 1$



So $I = (-1, 1)$