Math 20580 Name:______________________________

Final Exam Instructor:______________________________

December 13, 2023 Section:______________________________

Calculators are NOT allowed. You will be allowed 2 hours to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1.	a	~~b~~	c	d	e	11.	a	b	~~c~~	d	e
2.	a	b	c	~~d~~	e	12.	a	b	c	d	~~e~~
3.	a	b	c	d	~~e~~	13.	a	~~b~~	c	d	e
4.	a	b	~~c~~	d	e	14.	a	~~b~~	c	d	e
5.	a	b	c	d	~~e~~	15.	a	b	c	~~d~~	e
6.	a	b	c	~~d~~	e	16.	a	~~b~~	c	d	e
7.	a	b	~~c~~	d	e	17.	a	b	~~c~~	d	e
8.	~~a~~	b	c	d	e	18.	a	b	c	d	~~e~~
9.	a	~~b~~	c	d	e	19.	a	b	~~c~~	d	e
10.	~~a~~	b	c	d	e	20.	~~a~~	b	c	d	e

1. Let W be the column space of a 23×2023 matrix of rank 3. What is the dimension of $W^\perp$ (the orthogonal complement of W)?

(a) 3 (b) 20 (c) 23 (d) 2020 (e) 2023

W subspace of $\mathbb{R}^{23}$ of dimension 3

$\Rightarrow \dim W^\perp = 23-3$
$= 20$

2. Let M be the following matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of the following are eigenvalues of M?

I. 0 II. 1 III. 2 IV. 3

(a) I, II, and IV only (b) I, II, and III only (c) II, III, and IV only

(d) I and IV only (e) II and IV only

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = -\lambda^3 + 3\lambda^2$$
$$= \lambda^2(3-\lambda)$$

eigenvalues $\lambda = 0$
$\lambda = 3$

3. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$, the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 1 & -4 & 2 & 1 \\ 3 & -10 & 4 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & -4 & 2 & 1 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc:cc} 1 & -4 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

4. Describe the implicit solution of the exact equation

$$\underbrace{(e^x \sin(y) + 2y)}_{M}dx + \underbrace{(2x + e^x \cos(y))}_{N}dy = 0.$$

(a) $e^x \sin(y) = x^2 + y^2 + C$ (b) $e^x \sin(y) + 2xy + g(y)$ (c) $2xy + e^x \sin(y) = C$

(d) $y = \dfrac{e^x + x^2}{\sin(y)} + C$ (e) $e^x \tan(y) = C$

$$f(x,y) = \int M(x,y)\,dx = e^x \sin y + 2xy + g(y)$$

$$\frac{\partial f}{\partial y} = e^x \cos y + 2x + g'(y) = \overbrace{2x + e^x \cos y}^{N}$$

$\Rightarrow g'(y) = 0$, can take $g(y) = 0$

so $f(x,y) = e^x \sin y + 2xy$

Implicit soln.: $f(x,y) = C$

5. Describe the largest interval where a solution for the following initial value problem is guaranteed to exist:

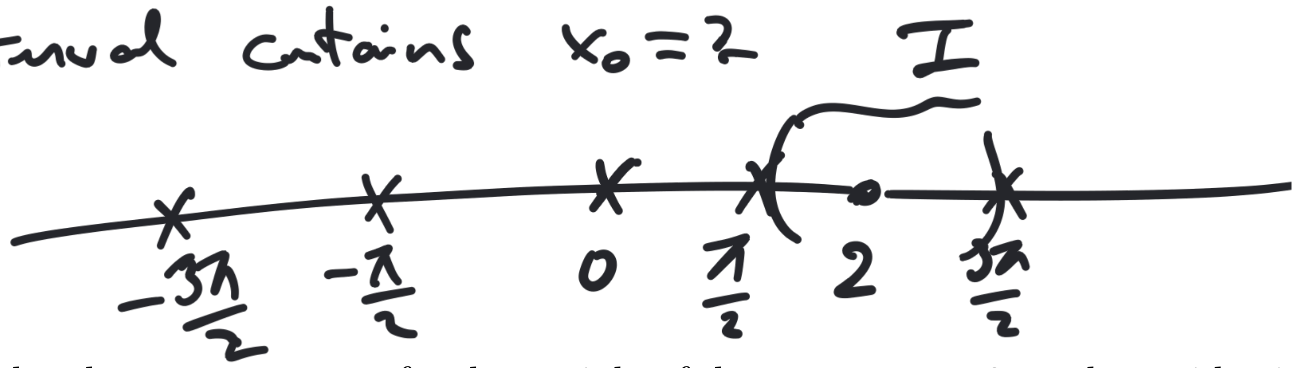
$$\begin{cases} (\cos x)y'' + y' + (\ln|x|)y = x^2 \\ y(2) = 1, \;\; y'(2) = -1 \end{cases}$$

(a) $(0, \pi)$ (b) $(-\infty, \infty)$ (c) $(-\pi/2, \pi/2)$ (d) $(0, \infty)$ (e) $(\pi/2, 3\pi/2)$

$\cos x \neq 0$, so $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$\ln|x|$ continuous so $x \neq 0$

Interval contains $x_0 = 2$



6. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2, and consider its basis $\mathcal{B} = \{1, 2-t, (2-t)^2\}$. The coordinate vector of $3t^2 - 8t + 6$ relative to the basis $\mathcal{B}$ is:

(a) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ -8 \\ 6 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 6 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \leftarrow -c_2 - 4c_3 = -8 \Rightarrow \boxed{c_2 = -4} \\ \leftarrow \boxed{c_3 = 3} \end{array}$$

$c_1 + 2c_2 + 4c_3 = 6$

$\Rightarrow \boxed{c_1 = 2}$

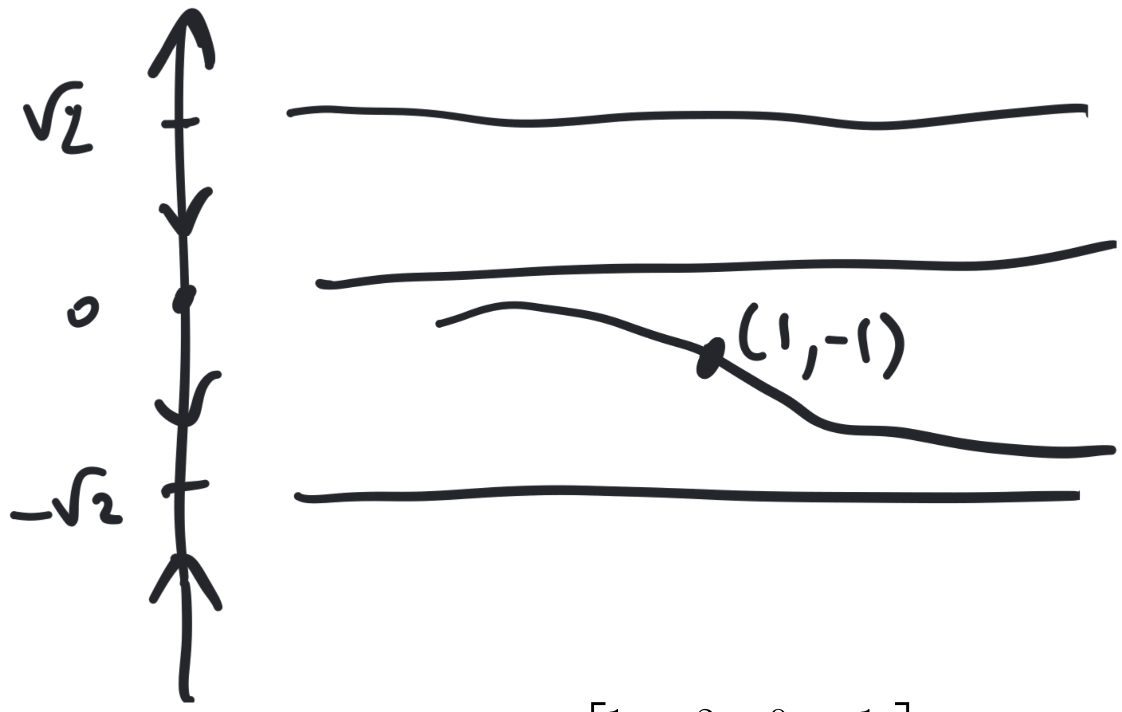
7. Consider the solution $y(x)$ of the autonomous equation

$$\frac{dy}{dx} = y^2 \cdot (y^2 - 2)$$

satisfying the initial condition $y(1) = -1$. Compute $\lim_{x\to\infty} y(x)$.

(a) $\sqrt{2}$ (b) $-\infty$ (c) $-\sqrt{2}$ (d) 0 (e) ∞

critical points $y^2(y^2-2)=0$ $y = 0, \pm\sqrt{2}$



8. Given the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, determine its nullity and rank.

(a) nullity$(A) = 1$, rank$(A) = 3$ (b) nullity$(A) = 2$, rank$(A) = 3$
(c) nullity$(A) = 0$, rank$(A) = 5$ (d) nullity$(A) = 1$, rank$(A) = 4$
(e) nullity$(A) = 2$, rank$(A) = 2$

$$A \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

rank $= 3$

nullity $= 4 - 3 = 1$

$$\xrightarrow[R_5 \to R_4 - \frac{1}{6}R_3]{R_4 \to R_4 + \frac{1}{6}R_3} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

9. The matrix $A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & -3 & -1 & 9 \end{bmatrix}$ has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

For which value of the parameter t is the vector $[1 \quad -1 \quad t \quad 0]$ in the row space of A?

(a) 2 (b) $\frac{7}{2}$ (c) 0 (d) -7 (e) -2

$$x_1 \cdot [1 \ 0 \ 0 \ 14] + x_2 [0 \ 1 \ 0 \ 7] + x_3 [0 \ 0 \ 1 \ -2]$$

$$= [x_1 \quad x_2 \quad x_3 \quad 14x_1 + 7x_2 - 2x_3]$$

$$= [1 \quad -1 \quad t \quad 0]$$

implies $x_1 = 1$, $x_2 = -1$, $x_3 = t$

$14x_1 + 7x_2 - 2x_3 = 0$

$7 - 2t = 0$

$t = 7/2$

10. If $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then b_{21} is:

(a) 0 (b) $-1/2$ (c) -2 (d) $3/2$ (e) 2

$$b_{21} = \frac{C_{12}}{\det A}$$

$$C_{12} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$= \frac{0}{\det A} = 0$$

11. Find the solution of the initial value problem

$$\begin{cases} y'' - 6y' + 9y = 0, \\ y(0) = 0, \ y'(0) = 2. \end{cases}$$

(a) $2e^{-3t} - 2$ (b) $e^{3t} - e^{-3t}$ (c) $2te^{3t}$ (d) $-6e^{3t} + 3$ (e) $-2e^{t} + e^{-2t}$

$m^2 - 6m + 9 = 0$ double root $m = 3$

FSS $= \{ e^{3t}, te^{3t} \}$, $y = (c_1 + c_2 t) e^{3t}$

$y(0) = 0 \Rightarrow \boxed{c_1 = 0} \Rightarrow y = c_2 te^{3t}$

$\Rightarrow y' = c_2 (1 + 3t) e^{3t}$

$y'(0) = 2$ $\Big\}$ $\boxed{c_2 = 2}$

so

$\underline{y = 2te^{3t}}$

12. Find the Wronskian $W(f_1, f_2, f_3)$ where $f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = 1/x$.

(a) x^2 (b) 0 (c) $\dfrac{x^2 + x^3 + 1}{x}$ (d) $\dfrac{x^2}{2} + \dfrac{x^3}{3} + \ln(x)$ (e) $\dfrac{6}{x}$

$$\begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} = x \begin{vmatrix} 2x & -x^{-2} \\ 2 & 2x^{-3} \end{vmatrix} - 1 \cdot \begin{vmatrix} x^2 & x^{-1} \\ 2 & 2x^{-3} \end{vmatrix}$$

$$= x \cdot (4x^{-2} + 2x^{-2}) - (\cancel{2x^{-1} - 2x^{-1}})$$

0

$$= \frac{6}{x}$$

13. Consider the differential equation $y'' + 4y = 4\sin(2x)$. Use the method of undetermined coefficients to find a particular solution.

(a) Ae^{4x} (b) $-x\cos(2x)$ (c) $e^{2x}\sin(2x)$ (d) $-2\cos(2x)$

(e) $x\sin(2x) + \cos(2x)$

$y'' + 4y = 0 \to m^2 + 4 = 0$ roots $m_{1/2} = \pm 2i$

$\Rightarrow FSS = \{\cos(2x), \sin(2x)\}$, $Y = x(A\cos 2x + B\sin 2x)$

$Y' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$

$Y'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$

$\Rightarrow 4B\cos 2x - 4A\sin 2x = 4\sin 2x \Rightarrow \boxed{B = 0, A = -1}$

So $Y = -x\cos(2x)$

14. Find the solution of the initial value problem

$$\begin{cases} y' = \dfrac{2x}{y + x^2 y} \\ y(0) = 1 \end{cases}$$

(a) $x^2\ln(x) + 1$ (b) $\sqrt{2\ln(x^2+1)+1}$ (c) $(x-1)^2$ (d) $\ln(x^2+1)+1$

(e) there is no solution

$\frac{dy}{dx} = \frac{2x}{y(x^2+1)} \Leftrightarrow \int y\,dy = \int \frac{2x}{1+x^2}\,dx$

$\Leftrightarrow \begin{cases} \frac{y^2}{2} = \ln(1+x^2) + C \\ y(0) = 1 \end{cases}$

$\boxed{\frac{1}{2} = C}$

$\Rightarrow y^2 = 2\ln(1+x^2) + 2\cdot\frac{1}{2}$

$= 2\ln(1+x^2) + 1$

$\Rightarrow y = \sqrt{2\ln(1+x^2)+1}$

(positive $\sqrt{\ }$ because $y(0) = 1 > 0$)

15. Which of the following matrices is similar to $A = \begin{bmatrix} -3 & 41 \\ -1 & 7 \end{bmatrix}$.

(a) $\begin{bmatrix} -1 & -3 \\ 7 & 41 \end{bmatrix}$ (b) $\begin{bmatrix} 20 & 21 \\ -4 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}$

eigenvalues : $\det \begin{bmatrix} -3-\lambda & 41 \\ -1 & 7-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 20$

$= (\lambda-2)^2 + 16 = 0$

$\Rightarrow \lambda = 2 \pm 4i \Rightarrow$ A similar to $\begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$

16. Find the matrix R in the QR decomposition of $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, provided that

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(a) $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{8} \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(d) $\begin{bmatrix} \sqrt{2} & -\sqrt{3} & -1 \\ 0 & \sqrt{3} & \sqrt{2} \\ 0 & 0 & \sqrt{8} \end{bmatrix}$ (e) none of the above

$$Q^TA = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

17. Which of the following describes the least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ only (b) $\begin{bmatrix} 1/2 \\ -3 \end{bmatrix}$ only (c) $\begin{bmatrix} 5/6 \\ 1/3 \end{bmatrix}$ only

(d) infinitely many solutions (e) no solutions

$A^TA = \begin{bmatrix} 6 & 6 \\ 6 & 27 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$

$\left[\begin{array}{cc|c} 6 & 6 & 7 \\ 6 & 27 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & 6 & 7 \\ 0 & 21 & 7 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 7/6 \\ 0 & 1 & 1/3 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & 5/6 \\ 0 & 1 & 1/3 \end{array}\right]$

18. Which formula describes the general solution of the differential equation

$$ty'' - (1+t)y' + y = 0, \ t > 0$$

given the fact that $y_1(t) = e^t$ is a solution of this equation.

(a) $c_1 + c_2e^t$ (b) $c_1e^t + c_2te^t$ (c) $c_1e^t + c_2\ln(t+1)$

(d) $c_1e^t + c_2e^{-t}$ (e) $c_1e^t + c_2(t+1)$

$y'' - \frac{1+t}{t}y' + \frac{1}{t}y = 0$

$y_2 = e^t \int \frac{e^{\int \frac{1+t}{t}dt}}{e^{2t}} dt$

$\Rightarrow y_2 = e^t \int \frac{e^{\ln t + t}}{e^{2t}} dt = e^t \cdot \int te^{-t} dt = e^t \cdot (-te^{-t} - e^{-t})$

$= -t-1 = (-1)\cdot(t+1)$

19. Consider the differential equation $x^2y'' - 2y = 3x^2 - 1$. The functions

$$y_1 = x^2 \quad \text{and} \quad y_2 = x^{-1}$$

form a fundamental set of solutions for the associated homogeneous equation. Which of the following describes a particular solution of the non-homogeneous equation?

(a) $x^2 - \dfrac{1}{x}$ (b) $c_1x^2 + c_2x^{-1}$ (c) $x^2\ln(x) + \dfrac{1}{2}$ (d) $\dfrac{x^2}{3x^2-1}$

(e) $\dfrac{x^3}{3} + \ln(x)$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -3$$

$$Y = u_1y_1 + u_2y_2$$

$$y'' - \frac{2}{x^2}y = \underbrace{3 - x^{-2}}_{g(x)}$$

$$u_1 = \int \frac{-x^{-1}(3 - x^{-2})}{-3}dx = \ln x + \frac{x^{-2}}{6}$$

$$u_2 = \int \frac{x^2(3 - x^{-2})dx}{-3} = \frac{x - x^3}{3}$$

$$Y = x^2\ln x + \frac{1}{6} + \frac{1 - x^2}{3}$$

$$= x^2\ln x + \frac{1}{2} - \frac{x^2}{3}$$

← solution of homogeneous eqn., can ignore

20. Find the general solution of the equation

$$(4 + x^2)\frac{dy}{dx} + 2xy = 4x.$$

(a) $\dfrac{2x^2 + C}{4 + x^2}$ (b) $\ln(4 + x^2) + C$ (c) $\dfrac{C}{4 + x^2}$ (d) $\dfrac{2x}{4 + x^2} + C$

(e) cannot be found explicitly using methods we learned

$$\frac{dy}{dx} + \frac{2x}{4 + x^2}\cdot y = \frac{4x}{4 + x^2}$$

$$\mu(x) = e^{\int \frac{2x}{4+x^2}dx} = e^{\ln(4+x^2)} = 4 + x^2$$

$$y = \frac{\int (4+x^2)\cdot\frac{4x}{4+x^2}dx}{4+x^2} = \frac{2x^2 + c}{4+x^2}$$