Math 20580 Name:____________________
Midterm 2 Instructor:____________________
March 9, 2023 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c ~~d~~ e
2. a ~~b~~ c d e
3. a ~~b~~ c d e
4. a ~~b~~ c d e
5. a b ~~c~~ d e
6. a b c d ~~e~~
7. ~~a~~ b c d e
8. a b c ~~d~~ e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Suppose that A and B are 3×3 matrices such that $\det(A) = 3$ and $\det(B) = -2$. What is $\det(3B^T A^{-1} B)$?

   (a) -36 (b) 0 (c) 4 (d) 36 (e) none of the above

$$3^3 \cdot \det(B) \cdot \frac{1}{\det A} \cdot \det B$$

$$= 27 \cdot (-2) \cdot \frac{1}{3} \cdot (-2) = 36$$

2. What are the eigenvalues of the matrix $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$?

   (a) -4, 0 (b) -3, -1 (c) -2, 1 (d) 1, 2 (e) none of the above

$$0 = \det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$= \lambda^2 + 4\lambda + 3$$

$$= (\lambda+1)(\lambda+3)$$

So $\lambda = -1$ and $\lambda = -3$

3. The vector $\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 0 & -1 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -1 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) -4 (b) -2 (c) 0 (d) 2 (e) 4

$$\begin{bmatrix} 0 & -1 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = (-2)\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$

4. Suppose that $T : \mathcal{P}_2 \to M_{2,2}$ is a linear transformation. Which of the following statements are always true? (Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2. Also recall that rank(T) is the dimension of the range of T, and nullity(T) is the dimension of the kernel of T.)

I. rank(T) + nullity(T) = 4.

II. T is one-to-one if and only if nullity$(T) = 0$. ✓

III. The range of T is a subspace of $\mathcal{P}_2$.

(a) I only (b) II only (c) I, III only (d) II, III only (e) none of them

I. rank (T) + nullity $(T) = \dim P_2 = 3$

II. one-to-one $\iff \ker T = \{\vec{0}\}$

$\iff$ nullity$(T) = 0$

III. Range is a subspace of $M_{2,2}$

5. Recall that $M_{2,2}$ is the vector space of 2×2 matrices. Consider the linear transformation

$$T : M_{2,2} \to M_{2,2}, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2b+c & b-c+d \\ -a-3c+2d & a+3b+d \end{bmatrix}.$$

Which of the following vectors is in the kernel of T?

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$\downarrow T$ $\quad\downarrow\quad\downarrow\quad\downarrow\quad\downarrow$

$\begin{bmatrix} 0 & -1 \\ \cdot & \cdot \end{bmatrix}$ $\begin{bmatrix} 2 & \cdot \\ \cdot & \cdot \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & -2 \\ \cdot & \cdot \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix}$

✗ ✗ ✓ ✗ ✗

6. Which of the following statements are always true for an $n \times n$ matrix A?

I. If A is invertible, then 0 is not an eigenvalue of A. ✓

II. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable. ✓

III. Every matrix similar to A has the same characteristic polynomial as A. ✓

(a) I only (b) I, II only (c) I, III only (d) II, III only (e) I, II, III

7. Consider the linear system

$$\begin{cases} x_1 + x_2 = 3, \\ x_1 - x_2 = 2. \end{cases}$$

According to Cramer's rule, what is x_2?

(a) $\dfrac{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (b) $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ (c) $\dfrac{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (d) $\dfrac{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (e) $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$

$A\vec{x} = \vec{b}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $x_2 = \dfrac{|A_2(\vec{b})|}{|A|}$

8. Recall that $M_{2,2}$ is the vector space of 2×2 matrices. Consider the function

$$T : M_{2,2} \to \mathbb{R}^2, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 + b^2 \\ c^2 + d^2 \end{bmatrix}.$$

Which of the following statements are true?

I. T is a linear transformation.

II. T is not a linear transformation because $T(\vec{0}) \neq \vec{0}$.

III. T is not a linear transformation because there exist A in $M_{2,2}$ and a scalar k such that $T(kA) \neq kT(A)$.

(a) none of them (b) I only (c) II only (d) III only (e) II, III only

II. $T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

III. $T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

for $k=2$, $T\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} \neq k \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the bases

$$\mathcal{B} = \{1 - x, x - x^2, x^2\} \quad \text{and} \quad \mathcal{C} = \{1 - x + x^2, 1 + 3x, 2 - x - 2x^2\}$$

of $\mathcal{P}_2$ (the vector space of polynomials of degree at most 2 in the variable x).

(a) Find the change-of-basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

Use $\mathcal{E} = \{1, x, x^2\}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & -1 & 3 & -1 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & -1 & 0 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 4 & -1 \end{matrix}}_{\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}}$$

(b) Suppose that $p(x)$ is a vector in $\mathcal{P}_2$ with $[p(x)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. What is $[p(x)]_{\mathcal{B}}$?

$$[p(x)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \cdot [p(x)]_{\mathcal{C}}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

10. Recall that $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2 in the variable x. Consider the linear transformation

$$T : \mathcal{P}_2 \to \mathcal{P}_2, \quad T(p(x)) = p(x) - (1+x)p'(x),$$

where $p'(x)$ is the derivative of $p(x)$.

(a) Verify that T can be expressed more explicitly as

$$T(\underbrace{a + bx + cx^2}_{p(x)}) = (a - b) - 2cx - cx^2.$$

$p'(x) = b + 2cx$

$\Rightarrow T(p(x)) = a + bx + cx^2 - (1+x)(b + 2cx) = (a-b) - 2cx - cx^2$

(b) Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis of $\mathcal{P}_2$. Find the matrix $[T]_{\mathcal{E}} = \underset{\mathcal{E} \leftarrow \mathcal{E}}{[T]}$ of T with respect to $\mathcal{E}$.

$$[T]_{\mathcal{E}} = \left[[T(1)]_{\mathcal{E}} \; [T(x)]_{\mathcal{E}} \; [T(x^2)]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Find a basis for the kernel of T and a basis for the range of T.

$$\text{Null} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \leftarrow x_1 = t \\ \\ \leftarrow x_3 = 0 \end{array}$$

$\uparrow$ $x_2 = t$ free

So

$$\text{Null } [T]_{\mathcal{E}} = \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\Rightarrow$ $\boxed{\text{Ker } T \text{ has basis } \{1+x\}}$

Col $[T]_{\mathcal{E}}$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \right\}$

$\Rightarrow$ Range (T) has basis $\{1, -2x - x^2\}$

11. Consider the matrix

$$A = \begin{bmatrix} 1 & t & -1 \\ 0 & 3 & t \\ 2 & 1 & -2 \end{bmatrix},$$

where t is some real number.

(a) Calculate the determinant of A. (Your answer may depend on t.)

$$\begin{vmatrix} 1 & t & -1 \\ 0 & 3 & t \\ 2 & 1 & -2 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1}{=\!=\!=} \begin{vmatrix} 1 & t & -1 \\ 0 & 3 & t \\ 0 & 1-2t & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & t \\ 1-2t & 0 \end{vmatrix} = -t(1-2t)$$

$$= -t + 2t^2$$

(b) Find all values of t such that A is invertible.

A invertible $\Leftrightarrow$ $\det A \neq 0$

$\Leftrightarrow$ $t(1-2t) \neq 0$

$\Leftrightarrow$ $t \neq 0$ and $t \neq \frac{1}{2}$

12. Let A be the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $\det(A - \lambda I) = (1-\lambda)^2(-2-\lambda)$.

(a) What are the eigenvalues of A?

$0 = (1-\lambda)^2(-2-\lambda)$

$\Rightarrow \lambda = 1$ or $\lambda = -2$

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$E_1 = \text{null}(A - I) = \text{null}\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$

$\downarrow$ RREF

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to x_1 = t - 2s$

$x_2 = t \quad x_3 = s$

$\Rightarrow E_1 = \left\{ \begin{bmatrix} t-2s \\ t \\ s \end{bmatrix} \right\} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{= \vec{v}_1}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{= \vec{v}_2} \right\} \leftarrow$ basis

$E_{-2} = \text{null}(A + 2I) = \text{null}\begin{bmatrix} 4 & -1 & 2 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}$

$\downarrow$ RREF

$E_{-2} = \text{Span}\left\{ \underbrace{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}_{= \vec{v}_3} \right\} \leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow x_1 = -t \\ \leftarrow x_2 = -2t \\ \end{matrix}$

$x_3 = t$

next page $\longrightarrow$

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$