Math 20580 Name:________________________

Midterm 2 Instructor:________________________

October 28, 2021 Section:________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a ~~b~~ c d e
2. a b c d ~~e~~
3. ~~a~~ b c d e
4. a b ~~c~~ d e
5. a b c ~~d~~ e
6. a b c ~~d~~ e
7. a ~~b~~ c d e
8. a ~~b~~ c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Assume that A and B are two 4×4 matrices with determinants $\det A = 2$, $\det B = 3$. Find the determinant $\det(2A^T B^2 A B^{-1})$.

   (a) 0 (b) 192 (c) -36 (d) 48 (e) cannot be determined.

$$\det(2A^T B^2 A B^{-1}) = 2^4 \det(A^T) \det(B)^2 \det(A) \det(B^{-1})$$

$$= 2^4 \cdot 2 \cdot 3^2 \cdot 2 \cdot \frac{1}{3}$$

$$= 192$$

2. What are the eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$?

   (a) 1,4 (b) 1,–2 (c) 2,1 (d) 3,0
   (e) none of the above.

$$\det(A - \lambda I) = (1-\lambda)(4-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3)$$

E-vals are: $\lambda = 2$, $\lambda = 3$

3. Let $M_{2,3}$ denote the vector space of 2×3 matrices. Which among the following subsets of $M_{2,3}$ is a subspace?

I. The set of all 2×3 matrices whose columns sum to $\begin{bmatrix}1\\0\end{bmatrix}$ $\vec{0} \notin S$

II. The set of all matrices whose entries are all non-negative. Not closed under mult. by neg. scalars.

III. $\left\{ \begin{bmatrix} t & t+s & s \\ 0 & s+2t & 0 \end{bmatrix} \middle| t,s\in\mathbb{R}\right\}$ $= \text{span}\left\{ \begin{bmatrix}1&1&0\\0&2&0\end{bmatrix}, \begin{bmatrix}0&1&1\\0&1&0\end{bmatrix}\right\}$

IV. The set of all matrices with a zero in the first row, second column.

(a) III and IV only (b) IV only (c) I, III, and IV only

(d) all of them (e) none of them.

$\underline{IV} = \text{span}\left\{ \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$

4. Which of the following statements is always true?

I. If two matrices are similar, then they have the same determinant.

II. If two matrices have the same characteristic polynomial, then they are similar.

III. If a matrix is diagonalizable, then it is invertible.

IV. If a matrix A is invertible, then zero is not an eigenvalue of A.

(a) III and IV only (b) I only (c) I and IV only

(d) all of them (e) none of them.

$\underline{II}$: $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ + $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ have same char. poly, but if they were similar, $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ would be diagonalizable, which it isn't.

$\underline{III}$: $O_{2,2}$ is diagonalizable but not invertible

5. Which of the following matrices has complex eigenvalue $4+2i$?

(I) $\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$ (II) $\begin{bmatrix} 4 & 2 \\ -2 & -4 \end{bmatrix}$ (III) $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ (IV) $\begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}$

(a) I and II only (b) IV only (c) III only (d) I and IV only (e) II and IV only

	Char poly	e-vals
I	$\lambda^2-8\lambda+20$	$4\pm 2i$
II	λ^2-12	$\pm 2\sqrt{3}$
III	$\lambda^2-8\lambda+12$	$2, 6$
IV	$\lambda^2-8\lambda+20$	$4\pm 2i$

6. Let $H = \text{span}\{\underset{p_1}{1}, \underset{p_2}{t^2+1}, \underset{p_3}{t^3+t^2+t}, \underset{p_4}{t^3+t-1}\}$, considered as a subspace of $\mathbb{P}_3$ (the vector space of all polynomials of degree at most 3). What is the dimension of H?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Two Methods

1) $t^3+t-1 = (t^3+t^2+t) + (-1)(t^2+1) + 0\cdot 1$, so

$H = \text{span}\{1, t^2+1, t^3+t^2+t\}$

The set $\{1, t^2+1, t^3+t^2+t\}$ is LI $\Rightarrow \dim H = 3$

2) Express everything in terms of

$\mathcal{B} = \{1, t, t^2, t^3\}$:

$$A = \begin{array}{c} \begin{matrix} [p_1]_{\mathcal{B}} & [p_2]_{\mathcal{B}} & [p_3]_{\mathcal{B}} & [p_4]_{\mathcal{B}} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$\dim H = rk(A)$

$$\xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow rk(A) = 3 = \dim(H)$$

$V \longrightarrow W$

7. Let $T\colon \mathbb{R}^8 \to \mathbb{R}^{12}$ be a linear transformation which is one-to-one. What is the dimension of the range of T?

(a) 12 (b) 8 (c) 4 (d) 0 (e) cannot be determined.

Rank-nullity: $\dim(\text{range}(T)) + \dim(\ker(T)) = \dim V = 8$

T one-to-one $\Longleftrightarrow \dim(\ker(T)) = 0$

$\Rightarrow \dim(\text{range}(T)) = 8$

8. The vector $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 3 & -7 & 0 \\ 0 & -18 & 0 \\ 0 & 6 & 0 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) 2 (b) -18 (c) 7 (d) 0 (e) 3

$$A \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -18 \\ -54 \\ 18 \end{bmatrix}$$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & \sqrt{91} & \pi \\ 1 & 1 & 0 & 11000 & 10 \\ 1 & 1 & 1 & 1 & \sin(8) \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

(a) Calculate the determinant of A. Explain how this computation implies that A is invertible. (**Hint:** The size of the matrix and the irrational entries should encourage you to be efficient in your computation.)

Row Reduction

R_2+3R_1, R_3-R_1, R_4-R_1

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & \sqrt{91} & \pi \\ 0 & 0 & -1 & 11000 & 10 \\ 0 & 0 & 0 & 1 & \sin(8) \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} =: B$$

$\boxed{\det(A) \neq 0, \text{ so invertible}}$

$$\det(A) = \det(B) = 1\cdot 3\cdot(-1)\cdot 1\cdot 2$$

$$\boxed{= -6}$$

Cofactor Expansion

$$\det(A) = (-1)^{5+5}\, 2\cdot \det\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 0 & 0 & \sqrt{91} \\ 1 & 1 & 0 & 11000 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= (-1)^{5+5}\cdot 2\cdot\left[(-1)^{1+3}\cdot 1\det\begin{bmatrix} -3 & 0 & \sqrt{91} \\ 1 & 1 & 11000 \\ 1 & 1 & 1 \end{bmatrix} + (-1)^{4+3}\cdot 1\det\begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & \sqrt{91} \\ 1 & 1 & 11000 \end{bmatrix}\right]$$

$$= 2\cdot\left(-3(1-\cancel{11000}) + \cancel{\sqrt{91}\cdot 0} - \left(\cancel{-\sqrt{91}} - \left((-3)\cancel{(11000)} - \cancel{\sqrt{91}}\right)\right)\right)$$

$$= 2(-3) = -6$$

(b) Compute the entry in the the 5th row and 5th column of A^{-1}.

$$(A^{-1})_{55} = \frac{(-1)^{5+5}}{\det(A)}\cdot\det(A_{55})$$

$$= \frac{1}{2\cdot\det(A_{55})}\cdot\det(A_{55}) = \boxed{\frac{1}{2}}$$

10. Consider the two ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}, \qquad \mathcal{C} = \left\{ \underset{\vec{c}_1}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{c}_2}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{c}_3}{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$ (recall that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \cdot [\vec{x}]_{\mathcal{C}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left[[\vec{c}_1]_{\mathcal{B}} \mid [\vec{c}_2]_{\mathcal{B}} \mid [\vec{c}_3]_{\mathcal{B}} \right] = \boxed{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}}$$

$$\vec{c}_1 = \vec{b}_1 \Rightarrow [\vec{c}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{c}_2 = \vec{b}_2 + 2\vec{b}_3 \Rightarrow [\vec{c}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{c}_3 = -\vec{b}_1 - \vec{b}_3 \Rightarrow [\vec{c}_3]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

(b) If $\vec{v}$ is a vector in $\mathbb{R}^3$ with $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine $[\vec{v}]_{\mathcal{B}}$ and $\vec{v}$.

$$\vec{v} = \vec{c}_1 + \vec{c}_2 + \vec{c}_3 = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$[\vec{v}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\vec{v}]_{\mathcal{C}} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

11. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

$\lambda = 1, -1, 2$ (A is upper triangular, so e-vals are entries on diagonal.)

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

null$(A - \lambda I)$ computations.

$$\lambda = 1: \begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \quad \begin{bmatrix} -1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(An answer; there is some freedom)

(c) Express A^4 in the form PEP^{-1}, where E is a diagonal matrix.

$$\boxed{E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}} = D^4$$

12. Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 in the variable x, and the linear transformation

$$T : \mathbb{P}_2 \to \mathbb{P}_2, \qquad T(p(x)) = \frac{\partial}{\partial x}\left(p(x+2)\right),$$

where $\frac{\partial}{\partial x}$ means taking the derivative with respect to x.

(a) Verify that T can be expressed more explicitly as

$$T(a_0 + a_1 x + a_2 x^2) = (a_1 + 4a_2) + 2a_2 x.$$

$$T(a_0 + a_1 x + a_2 x^2) = \frac{\partial}{\partial x}\left(a_0 + a_1(x+2) + a_2(x+2)^2\right)$$

$$= (a_1 + 4a_2) + (2a_2 x)$$

(b) Write down a basis $\mathcal{B}$ for $\mathbb{P}_2$. Find the matrix $[T]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ of T with respect to $\mathcal{B}$.

$$\mathcal{B} = \{1, x, x^2\}$$

$$[T]_{\mathcal{B}} = \left[\, [T(\vec{b}_1)]_{\mathcal{B}} \;\middle|\; [T(\vec{b}_2)]_{\mathcal{B}} \;\middle|\; [T(\vec{b}_3)]_{\mathcal{B}} \,\right] = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Find bases for the kernel and the range of T.

Set $A = [T]_{\mathcal{B}}$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for null(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\Rightarrow$ Basis for ~~ker~~ ker(T) $= \{1 + 0x + 0x^2\}$

Basis for col(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \right\}$ $\Rightarrow$ Basis for range(T) $= \{1 + 0x + 0x^2,\ 4 + 2x + 0x^2\}$