Math 20580
Midterm 3
April 20, 2023

Name: ______________________________
Instructor: ______________________________
Section: ______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b ~~c~~ d e
2. a b c ~~d~~ e
3. ~~a~~ b c d e
4. a ~~b~~ c d e
5. a b ~~c~~ d e
6. a ~~b~~ c d e
7. ~~a~~ b c d e
8. a b ~~c~~ d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Which of the following is an eigenvalue of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ corresponding to the eigenvector $\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}$?

   (a) $\lambda = 1 + 3i$ (b) $\lambda = i$ (c) $\lambda = 1 + i$ (d) $\lambda = 1$ (e) $\lambda = -1$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i-1 \\ i+1 \end{bmatrix}$$

$$= (i+1) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

2. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}$. What is the dimension of the orthogonal complement $W^\perp$ of W?

   (a) 3 (b) 0 (c) 2 (d) 1 (e) None

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}^\perp$$

So $W^\perp = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ has dimension one

3. Consider the orthogonal vectors $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The distance from the vector $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ to $W = \text{Span}\{\vec{w}_1, \vec{w}_2\}$ is:

(a) $\sqrt{6}$ (b) 1 (c) $\sqrt{2}$ (d) 0 (e) $\sqrt{3}$

$$\text{proj}_W(\vec{x}) = \frac{\vec{x}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \frac{\vec{x}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2$$

$$= \frac{2}{2}\vec{w}_1 + \frac{3}{3}\vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{perp}_W(\vec{x}) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix},$$

distance $= \sqrt{1^2+2^2+1^2} = \sqrt{6}$

4. The matrix $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ factors as $A = QR$, where $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ -1/\sqrt{2} & 2/3 \\ 0 & 1/3 \end{bmatrix}$ and R is:

(a) $\begin{bmatrix} 1/\sqrt{2} & 1/3 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sqrt{3} & 1 \\ 0 & -\sqrt{2} \end{bmatrix}$ (e) $\begin{bmatrix} 2/3 & 1/3 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 3 \end{bmatrix}$$

5. Let $y(x)$ be the unique solution of the initial value problem

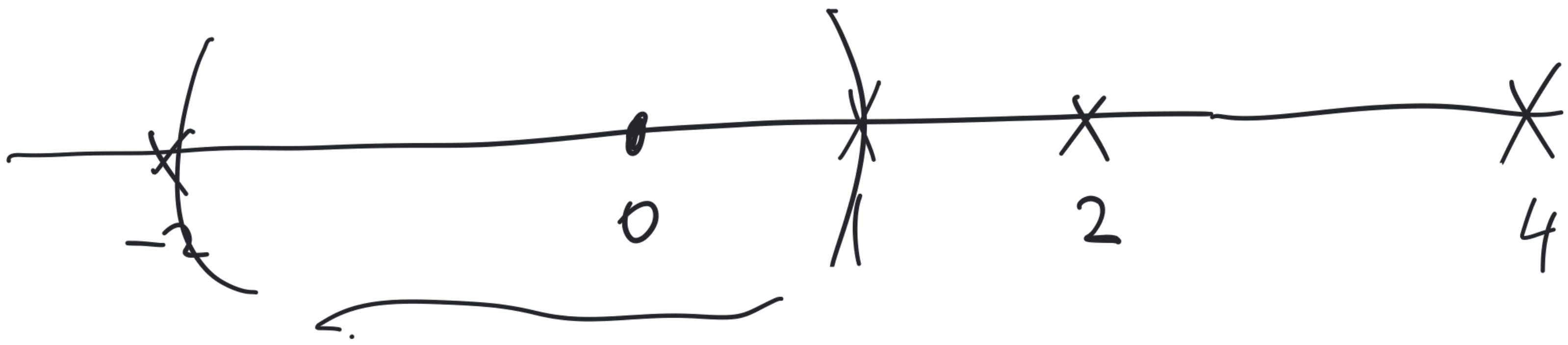
$x^2 \le 4$

$$\sqrt{4-x^2}\, y'' + \frac{x}{x^2+1}\, y' + 5y = \frac{1}{x^2-5x+4}, \quad y(0)=1,\ y'(0)=7.$$

$x \neq 1, 4$

What is the largest interval where $y(x)$ is defined?

(a) $x \geq 0$ (b) $-2 \leq x \leq 2$ (c) $-2 < x < 1$ (d) $-2 < x < 2$ (e) $x < 2$



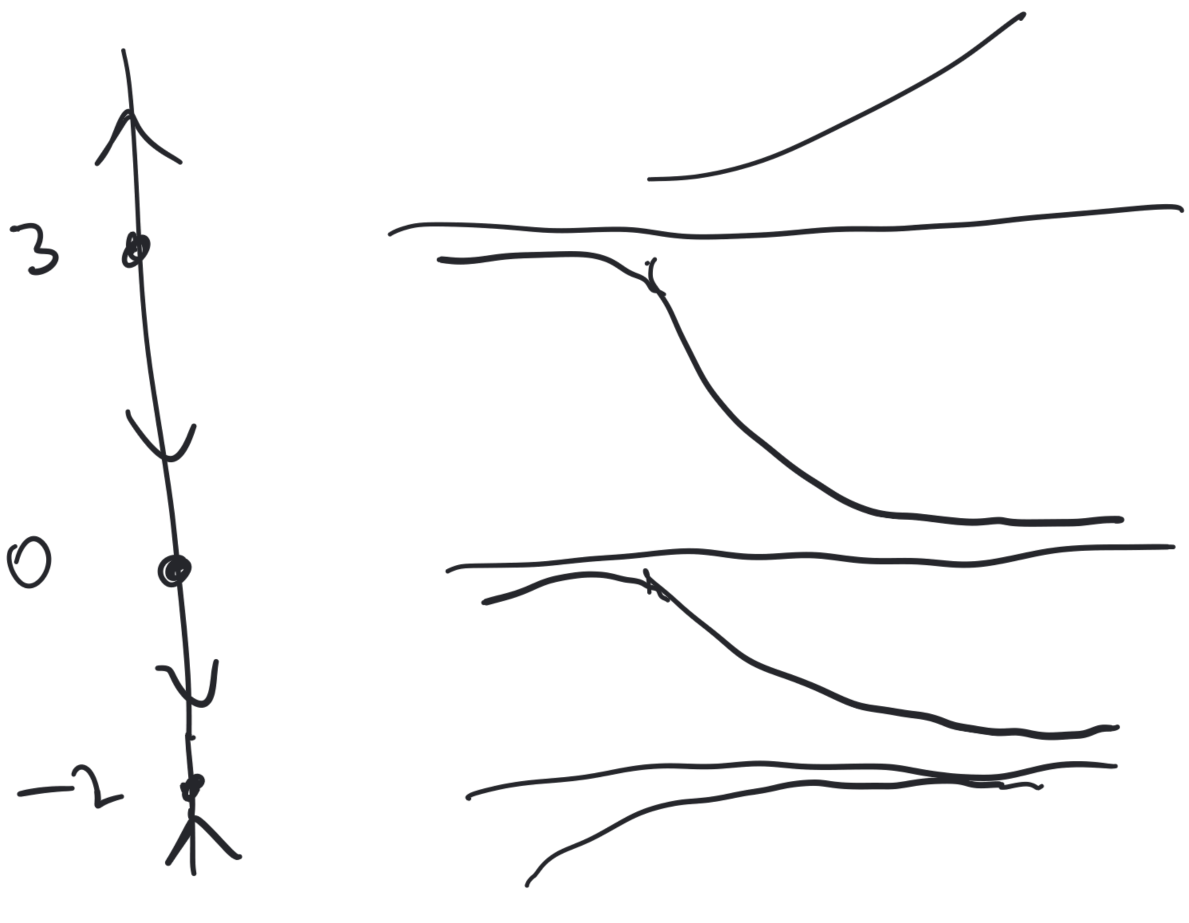
longest interval

6. Find all stable critical values (also known as stable equilibrium solutions) for the autonomous system

$$\frac{dy}{dx} = y^2(y-3)(y+2).$$

(a) $y=3,\ y=0,\ y=-2$ (b) $y=-2$ (c) $y=3,\ y=0$

(d) $y=0$ (e) $y=3$



only stable one

7. Solve the initial value problem

$P(x)$ $\quad J(x)$

$$\frac{dy}{dx} - \frac{3}{x}y = x^6, \qquad y(1) = \frac{5}{4}.$$

(a) $y = \dfrac{x^7}{4} + x^3$ (b) $y = \dfrac{x^7}{4} + x$ (c) $y = \dfrac{x^7}{10} + \dfrac{23}{10}x^{-3}$

(d) $y = \dfrac{x^{10}}{7} + \dfrac{31}{28}$ (e) $y = \dfrac{x^{10}}{7} + x^3$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = x^{-3}$$

$$y = \frac{\int x^{-3}\cdot x^6\,dx}{x^{-3}} = \left(\int x^3\,dx\right)\cdot x^3 = \left(\frac{x^4}{4} + C\right)\cdot x^3$$

$$y(1) = \frac{1}{4} + C = \frac{5}{4} \Rightarrow \boxed{C = 1} \Rightarrow y = \frac{x^7}{4} + x^3$$

8. Which of the following is a solution to the initial value problem?

$$\frac{dy}{dt} = \frac{ty^2}{1+t^2}, \qquad y(0) = 1.$$

(a) $\ln(y) = \dfrac{t^3}{3}$ (b) $\dfrac{y^3}{3} = \tan^{-1}(t) + \dfrac{1}{3}$ (c) $y = \dfrac{1}{1 - \frac{1}{2}\ln(1+t^2)}$

(d) $\ln(y) = \tan^{-1}(t)$ (e) $y = \dfrac{1}{1 - \tan^{-1}(t)}$

$$\int \frac{dy}{y^2} = \int \frac{t}{1+t^2}\,dt \qquad -\frac{1}{y} = \frac{1}{2}\ln(1+t^2) + C$$

$$t=0: \quad -1 = C$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{2}\ln(1+t^2) - 1 \Rightarrow \boxed{y = \frac{1}{1 - \frac{1}{2}\ln(1+t^2)}}$$

Part II: Partial credit questions (11 points each). Show your work.

9. (a) Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace

$$V = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} \right\} \text{ of } \mathbb{R}^4.$$

$\vec{v}_1 = \vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 0 \cdot \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

orthogonal Basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

(b) Find an orthonormal basis for V from the orthogonal basis found in part (a).

$\|\vec{v}_1\| = \sqrt{4} = 2$

$\|\vec{v}_2\| = \sqrt{4+1+1} = \sqrt{6}$

$\|\vec{v}_3\| = \sqrt{1+1+1+9} = \sqrt{12}$

orthonormal basis

$$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

10. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

$A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad A^T\vec{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

normal equation $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

Solution

$$\hat{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \end{bmatrix}$$

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

$$\hat{b} = A\hat{x}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

11. Consider the differential equation $\Big(\underbrace{e^y - \sin(x)}_{M}\Big)dx + \left(\underbrace{xe^y}_{N} - \frac{3}{y}\right) dy = 0.$

(a) Show that the equation is exact.

$M_y = e^y$ $\qquad\qquad$ $M_y = N_x$

so exact

$N_x = e^y$

(b) Find the general implicit solution and express it in the form $f(x, y) = c$.

$$f(x,y) = \int (e^y - \sin x)\,dx$$

$$= xe^y + \cos x \quad + g(y)$$

$$\frac{\partial f}{\partial y} = N \Rightarrow \quad xe^y + g'(y) = xe^y - \frac{3}{y}$$

$$\Rightarrow g'(y) = -\frac{3}{y} \Rightarrow g(y) = -3\ln y$$

So $\boxed{f(x,y) = xe^y + \cos x - 3\ln y}$

Solution

$f(x,y) = C$

(c) Find the implicit solution that satisfies the initial condition $y(0) = e$.

$x = 0, y = e$ gives

$$0 \cdot e^e + \cos 0 - 3\ln e = C$$

$$1 - 3 = C$$

$\boxed{C = -2}$ $\Rightarrow$ implicit solution $\boxed{xe^y + \cos x - 3\ln y = -2}$

12. Willy has a tank containing 10 gallons of milk which initially contains 1 pound of chocolate powder. The well-mixed chocolate milk in the tank is drained at a rate of 3 gallons per hour, and Willy pumps in chocolate milk with a concentration of 1 pound of chocolate powder per gallon at a rate of 3 gallons per hour.

(a) Set up an initial value problem for the amount $y(t)$ in pounds of chocolate powder in the tank after t hours.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) \qquad \boxed{y(0)=1}$$

$$= 3 \cdot 1 - 3 \cdot \frac{y}{10}$$

$$\boxed{\frac{dy}{dt} = \frac{3}{10}(10-y)}$$

(b) Solve the initial value problem to find an explicit formula for $y(t)$.

$$\int \frac{dy}{10-y} = \int \frac{3}{10}\, dt$$

$$-\ln(10-y) = \frac{3t}{10} + C \quad \overset{y(0)=1}{\Longrightarrow} \quad -\ln 9 = C$$

$$\Rightarrow \quad -\ln(10-y) = \frac{3t}{10} - \ln 9$$

$$\overset{\exp}{\Longrightarrow} \quad \frac{1}{10-y} = \frac{1}{9} e^{3t/10} \Rightarrow 10-y = 9e^{-3t/10} \Rightarrow \boxed{y = 10 - 9e^{-3t/10}}$$