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Math 20580 Tutorial – Worksheet 1

1. Circle which of the following matrices are in **row echelon form**?

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 7 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 5 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 & -1 & 5 \\ 2 & 8 & 16 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 7 & 22 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: (2 points - Completion) B and D are both in row echelon form.

2. Use Gaussian elimination to solve the following systems of linear equations. What is the rank of the associated matrices?

$$\text{(a)} \begin{cases} 2x + y - z = 1 \\ x + 2y - z = -1 \\ 3x + 2y + z = 9 \end{cases} \qquad \text{(b)} \begin{cases} x + y - z = 0 \\ 3x + y - 2z = 0 \\ 2x - 6y + 2z = 0 \end{cases}$$

Solution: (3 points - Completion)

(a) The augmented matrix for this system of linear equations is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & -1 \\ 3 & 2 & 1 & 9 \end{array}\right]$$

Using Gaussian elimination gives:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & -1 \\ 3 & 2 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 9 \end{array}\right] \xrightarrow{R_2 - 2R_1, R_3 - 3R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -3 & 1 & 3 \\ 0 & 2 & 4 & 12 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -3 & 1 & 3 \\ 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & -3 & 1 & 3 \end{array}\right] \xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 7 & 21 \end{array}\right] \xrightarrow{\frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus we can conclude that the original system of equations has the same solutions as the following system of equations:

$$\begin{cases} x + 2y - z = -1 \\ y + 2z = 6 \\ z = 3 \end{cases}$$

Use back-substitution to see that $y + 6 = 6$, i.e. $y = 0$. Again using back substitution we see that $x + 0 - 3 = -1$, i.e. $x = 2$. So we obtain the sole solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

(b) The augmented matrix for this system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & -6 & 2 & 0 \end{array}\right]$$

Using Gaussian elimination gives:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & -6 & 2 & 0 \end{array}\right] \xrightarrow{R_2-3R_1,\, R_3-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -8 & 4 & 0 \end{array}\right] \xrightarrow{R_3-4R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{-1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we can conclude that the original system of equations has the same solutions as the following system of equations:

$$\begin{cases} x + y - z = 0 \\ y - \frac{1}{2}z = 0 \end{cases}$$

Assign the parameter $z(t) = t$. Then using back-substitution we have that $y = \frac{1}{2}t$, and $x + \frac{1}{2}t - t = 0$, i.e. $x = \frac{1}{2}t$. Thus there are infinitely many solutions to the system of equations, and they are parametrised by t as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

3. Given the following augmented matrices corresponding to some systems of linear equations, determine how many solutions they have (**if any**). Also, if the corresponding linear system is consistent, determine its rank and the number of free variables it has.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 8 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & -3 & -6 & 0 & 1 \\ 0 & 1 & -1 & 4 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right]$

(a) $\left[\begin{array}{cccc|c} 1 & 5 & 2 & -5 & 1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & -1 & 1 & 3 \end{array}\right]$

Solution: (2 points - Completion)

(a)

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 8 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & -3 & -6 & 0 & 1 \\ 0 & 1 & -1 & 4 & 0 \end{array}\right] \xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 8 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 1 & -1 & 4 & 0 \end{array}\right]$$

The third row represents the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 10$, i.e. $0 = 10$. Thus there are no solutions and it is inconsistent.

(b)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right] \xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent. The rank of the augmented matrix is three and, since there are four variables, one of the variables must be free. Thus there are infinitely many solutions.

(c)

$$\left[\begin{array}{cccc|c} 1 & 5 & 2 & -5 & 1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{cccc|c} 1 & 5 & 2 & -5 & 1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 2 & 12 \end{array}\right]$$

This system is consistent. It has rank four and there are four variables. Thus there is a unique solution.

4. (a) Show that the following pair of matrices is **row equivalent**, i.e. find a sequence of elementary row operations converting one into the other.

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

(b) Do the same for the following pair.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 6 \\ -3 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 5 & 1 & 9 \end{bmatrix}$$

Hint: put the matrices in row echelon form.

Solution: (3 points - Completion)

(a) An example, starting with A is the following sequence: $R_2 + R_1$, $2R_1$, $R_1 \leftrightarrow R_2$.

(b) Perform the following row operations on the matrix A: $R_2 - 2R_1$, $R3 + 3R_1$, $R_3 - R_2$. This gives the row echelon form for A as the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Perform the following sequence of row operations on B: $R_2 \leftrightarrow R_1$, $R_2 - 2R_1$, $R_3 - 5R_1$, $R_3 - 4R_2$, $-R_2$, $R_1 - R_2$. This results in the same matrix as above. Since they have the same row echelon form they are row equivalent. To see this recall that elementary row operations are reversible. In particular we can perform the first sequence to get to C and then perform the reverse of the second sequence to get to B. I.e. here is a sequence that goes from A to B:

$R_2 - 2R_1$, $R3 + 3R_1$, $R_3 - R_2$, $R_1 + R_2$, $-R_2$, $R_3 + 4R_2$, $R_3 + 5R_1$, $R_2 + 2R_1$, $R_2 \leftrightarrow R_1$.

5. Consider the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 2h+k \\ 1 & 1 & 1 & 2k \\ 0 & 1 & 3 & h \end{array}\right]$$

Find what conditions, if any, must be placed on h and k to ensure that the corresponding system of equations is consistent.

Solution: (5 points) 1 point for attempting to use Gaussian elimination, 2 points for getting the correct row echelon form, 2 points for identifying that $h - k$ must be 0.

Let us first try to get to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 2h+k \\ 1 & 1 & 1 & 2k \\ 0 & 1 & 3 & h \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2h+k \\ 0 & -1 & -3 & k-2h \\ 0 & 1 & 3 & h \end{array}\right] \xrightarrow{R_3 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 2h+k \\ 0 & -1 & -3 & k-2h \\ 0 & 0 & 0 & k-h \end{array}\right]$$

Thus the system is consistent if and only if $k - h = 0$, i.e. $k = h$.