Name: Date: 11/09/2023

M20580 L.A. and D.E. Tutorial
Worksheet 10

1. Determine whether the statements are true or false, and justify your answer.

   - If A is a $m \times n$ the QR factorization of A always exists.
   - If $Ax = b$ is a consistent linear system, then $A^T Ax = A^T b$ is also consistent, with the same solution.
   - If $Ax = b$ is an inconsistent linear system, then $A^T Ax = A^T b$ is also inconsistent.

Solution:

- False. It is a necessary additional hypothesis that the columns of A are linearly independent.
- True. A solution to $A^T Ax = A^T b$ is a least squares solution to $Ax = b$. This is the equivalent to finding the projection of b onto the column space of A and is thus always consistent.
- False. Since again $A^T Ax = A^T b$ is always consistent.

2. Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ -1 & 2 \\ 4 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 17 \\ 11 \\ 6 \\ 3 \end{bmatrix}.$$

(a) Find the least squares solution $\overline{x}$ to the equation $Ax = \mathbf{b}$.

(b) Find the vector in the column space of A which is closest to $\mathbf{b}$. (Hint: recall that when the columns for A are linearly independent, then $\text{proj}_{\text{col}(A)}v = A(A^TA)^{-1}A^Tv$ which from (a) is $A\overline{x}$.)

Solution: (a) Recall that if $\overline{x}$ is the least squares solution, then $A^TA\overline{x} = A^Tb$.

We first compute

$$A^TA = \begin{bmatrix} 22 & 0 \\ 0 & 14 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 11 \\ 42 \end{bmatrix}.$$

Solving for $A^TA\overline{x} = A^Tb$ gives

$$\overline{x} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$$

(b) Note that

$$\begin{bmatrix} -1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix} = 0 + 6 - 2 - 4 = 0$$

so the two columns for A are already orthogonal. Thus $\text{proj}_{\text{col}(A)}\mathbf{b} = \text{proj}_{c_1}\mathbf{b} + \text{proj}_{c_2}\mathbf{b}$ where c_1 and c_2 are the two columns of A as column vectors.

$$\begin{aligned} \text{proj}_{\text{col}(A)}\mathbf{b} = & \text{proj}_{c_1}\mathbf{b} + \text{proj}_{c_2}\mathbf{b} = \frac{c_1 \cdot \mathbf{b}}{c_1 \cdot c_1}c_1 + \frac{c_2 \cdot \mathbf{b}}{c_2 \cdot c_2}c_2 \\ = & \frac{11}{22}c_1 + \frac{42}{14}c_2 \\ = & \frac{1}{2}c_1 + 3c_2 \\ = & \begin{bmatrix} -1/2 \\ 10 \\ 11/2 \\ -1 \end{bmatrix}. \end{aligned}$$

As per the hint we see that this is the same thing as $A\overline{x}$.

3. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix}.$$

(a) Find the QR factorization for A.

(b) Use the QR factorization for A to find the least squares solution to $Ax = \mathbf{b}$.

Solution: (a) Compute the QR factorization as

$$QR = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \\ -1/\sqrt{2} & 2/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 3 \end{bmatrix}.$$

(b) Next recall that the least squares solution $\overline{x}$ is given by $R^{-1}Q^T\mathbf{b}$. Using this we compute:

$$\overline{x} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} \\ \sqrt{2}/3 \end{bmatrix}$$

4. Find an explicit solution for the separable differential equation

$$3x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y(x + 4x^2)$$

with the initial condition $y(1) = 2$.

Solution: First separate the variables:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3x} + \frac{4}{3}.$$

Next we integrate with respect to x to get:

$$\ln(y) = \frac{1}{3}\ln(x) + \frac{4x}{3} + C.$$

Exponentiating both sides gives

$$y = e^{1/3\ln(x)+4/3x+C}.$$

Split up the right hand side as $e^{1/3\ln(x)}e^{4/3x+C}$ and observe that

$$e^{1/3\ln(x)} = \left(e^{\ln(x)}\right)^{1/3} = x^{1/3} = \sqrt[3]{x}.$$

Thus we can simplify this to get

$$y = \sqrt[3]{x}e^{4/3x+C}.$$

Plugging in $x = 1$ and $y = 2$ gives $2 = e^{4/3+C}$. Taking log of both sides gives $\ln(2) = 4/3 + C$ and thus $C = \ln(2) - 4/3$. Putting it all together we obtain

$$y = \sqrt[3]{x}e^{4/3x+\ln(2)-4/3}.$$

Again we can simplify by observing that

$$e^{4/3x+\ln(2)-4/3} = e^{\ln(2)}e^{4/3x-4/3} = 2e^{4/3x-4/3}.$$

Thus our final answer is

$$y = 2\sqrt[3]{x}e^{4/3x-4/3}.$$