Name: Date: 11/16/2023

M20580 L.A. and D.E. Tutorial
Worksheet 11

1. (a) Solve the separable differential equation: $dx + e^{4x} dy = 0$
   (b) Solve the linear differential equation: $y' + 3x^2 y = x^2$.
   (c) Check if the differential equation is exact, then solve: $(2x-1)dx + (3y+7)dy = 0$

Solution:

(a) From $dy = -e^{-4x}dx$ we obtain $y = \frac{1}{4}e^{-4x} + c$.

(b) An integrating factor is $e^{\int 3x^2 dx} = e^{x^3}$ so that $\frac{d}{dx}\left[e^{x^3}y\right] = x^2 e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$.

(c) Let $M = 2x - 1$ and $N = 3y + 7$ so that $M_y = 0 = N_x$. From $f_x = 2x - 1$ we obtain $f = x^2 - x + h(y)$, $h'(y) = 3y + 7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

2. For each of the following differential equations, first check if it is linear, then solve it using an appropriate method.

   (a) $\dfrac{dy}{dt} = ty^2 \cos t, \qquad y(0) = 1$

   (b) $t\dfrac{dy}{dt} = t^2 + y, \qquad y(1) = -2 \qquad \text{for } t > 0,$

   (c) $y' = 2ty + 3t^2 e^{t^2},$

   (d) $t^2 \dfrac{dy}{dt} + ty = 1$, assuming $t > 0$.

Solution:

(a) The DE is **not linear**. Using the method of separation of variables, we have:

$$\begin{aligned}
&\frac{dy}{dt} = ty^2 \cos t \\
\implies &\frac{dy}{y^2} = t\cos t \; dt \\
\implies &\int y^{-2} dy = \int t \cos t \; dt \\
&\qquad \text{Integration by parts:} \\
&\qquad\qquad u = t \implies du = dt \\
&\qquad\qquad dv = \cos t \; dt \implies v = \sin t \\
\implies &\frac{y^{-1}}{-1} = t \sin t - \int \sin t \; dt \\
\implies &-y^{-1} = t \sin t - (-\cos t) + C \\
\implies &-y^{-1} = t \sin t + \cos t + C.
\end{aligned}$$

Since $y(0) = 1$, we have

$$-1^{-1} = 0 \sin 0 + \cos 0 + C \implies \qquad C = -1^{-1} - \cos 0 = -1 - 1 = -2.$$

Therefore,

$$\begin{aligned}
&-y^{-1} = t \sin t + \cos t - 2 \\
\implies &y^{-1} = 2 - t \sin t - \cos t \\
\implies &\boxed{y = \frac{1}{2 - t\sin t - \cos t}}.
\end{aligned}$$

(b) The DE is **linear**. We have

$$\begin{aligned} t\frac{dy}{dt} &= t^2 + y \\ \implies t\frac{dy}{dt} - y &= t^2 \\ \implies \frac{dy}{dt} - t^{-1}\cdot y &= t \end{aligned}$$

Multiplying both sides of the equation by the following integrating factor

$$I = e^{\int -t^{-1}dt} = e^{-\ln|t|} = e^{-\ln t} \qquad (|t| = t \text{ as } t > 0)$$
$$I = t^{-1}. \qquad (\text{Note: } e^{-\ln x} \text{ is NOT } -x)$$

We then have

$$\begin{aligned} t^{-1}\frac{dy}{dt} - t^{-2}y &= 1 \\ \implies (t^{-1}y)' &= 1 \\ \implies t^{-1}y &= \int 1\cdot dt \\ \implies t^{-1}y &= t + C \\ \implies y &= t^2 + Ct. \end{aligned}$$

Since $y(1) = -2$, we have

$$-2 = 1^2 + C\cdot 1 \implies C = -2 - 1 = -3.$$

Therefore, $\boxed{y = t^2 - 3t}$.

(c) The DE is **linear**. We have

$$\begin{aligned} y' &= 2ty + 3t^2e^{t^2} \\ \implies y' - 2t\cdot y &= 3t^2e^{t^2}. \end{aligned}$$

We multiply both sides of the equation by the following integrating factor

$$I = e^{\int(-2t)dt} = e^{-t^2}.$$

We then have

$$\begin{aligned} y'e^{-t^2} - 2te^{-t^2}y &= 3t^2 \\ \implies (ye^{-t^2})' &= 3t^2 \\ \implies ye^{-t^2} &= \int 3t^2dt \\ \implies ye^{-t^2} &= t^3 + C \\ \boxed{y = e^{t^2}(t^3 + C)}, \end{aligned}$$

where C is an arbitrary constant.

(d) The DE is **linear**. We have

$$t^2\frac{dy}{dt} + ty = 1$$
$$\implies \frac{dy}{dt} + t^{-1} \cdot y = t^{-2}.$$

We multiply both sides of the equation by the following integrating factor

$$I = e^{\int t^{-1}dt} = e^{\ln|t|} = |t| = t. \qquad (|t| = t \text{ for } t > 0)$$

We have

$$t\frac{dy}{dt} + y = t^{-1}$$
$$\implies (ty)' = t^{-1}$$
$$\implies ty = \int t^{-1}dt$$
$$\implies ty = \ln|t| + C$$
$$\implies ty = \ln t + C \qquad (|t| = t \text{ for } t > 0)$$
$$\boxed{y = t^{-1}\ln t + Ct^{-1}}.$$

3. Solve the given initial-value problem

$$x dx + (x^2 y + 4y) dy = 0, \quad y(4) = 0$$

This ODE is not exact, but you can make it exact by first finding an integrating factor.

Solution: We note that $(M_y - N_x)/N = 2x/(4 + x^2)$, so an integrating factor is $e^{-2\int x dx/(4+x^2)} = 1/(4 + x^2)$.

Let $M = x/(4 + x^2)$and $N = (x^2 y + 4y)/(4 + x^2) = y$, so that $M_y = 0 = N_x$. From $f_x = x/(4 + x^2)$ we obtain $f = \frac{1}{2}\ln(4 + x^2) + h(y)$, $h'(y) = y$, and $h(y) = \frac{1}{2}y^2$. A solution of the differential equation is

$$\frac{1}{2}\ln(4 + x^2) + \frac{1}{2}y^2 = c.$$

Multiplying both sides by 2 the last equation can be written as $e^{y^2}(x^2+4) = c_1$. Using the initial condition $y(4) = 0$ we see that $c_1 = 20$. A solution of the initial-value problem is $e^{y^2}(x^2 + 4) = 20$.