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M20580 L.A. and D.E. Tutorial
Worksheet 12

1. Solve the following homogeneous linear differential equations with constant coefficient.
   (a) $y'' - 8y' + 16y = 0$, with initial values $y(0) = 1/4$, $y'(0) = 0$.
   (b) $y'' + y' - 6y = 0$.
   (c) $y''' - 3y'' + 2y' = 0$.

Solution: (a) The auxiliary equation is $m^2 - 8m + 16 = (m-4)^2$. Thus we have a repeated real root, and we obtain the general solution

$$y = c_1 e^{4x} + c_2 x e^{4x}.$$

Since $y(0) = 1/4$ we see that $c_1 = 1/4$. We find

$$y' = 4c_1 e^{4x} + 4c_2 x e^{4x} + c_2 e^{4x}.$$

Since $y'(0) = 0$ we see that $4c_1 + c_2 = 0$ and thus $c_2 = -1$. So our unique solution is

$$y = \frac{1}{4}e^{4x} - xe^{4x}.$$

(b) The auxiliary equation is $m^2 + m - 6 = (m-2)(m+3)$. Thus the general solution is

$$y = c_1 e^{2x} + c_2 e^{-3x}.$$

(c) The auxiliary equation is $m^3 - 3m^2 + 2m = m(m^2 - 3m + 2) = m(m-1)(m-2)$. Since these are distinct roots a general solution is given by

$$y = c_1 + c_2 e^{x} + c_3 e^{2x}.$$

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2. Each of the following is a differential equation and a set of solutions for them. Determine whether they are linearly independent by calculating their Wronskian.

   (a) $y''' - y'' = 0,$ with solutions $\{0, x+1\}$.

   (b) $y''' = 0,$ with solutions $\{1, x, x^2\}$.

   (c) $y'' - 10y' + 16y = 0,$ with solutions $\{e^{2x}, e^{8x}\}$.

Solution: The Wronskian for each part is (a) 0, (b) 2, and (c) $6e^{10x}$. Thus the solutions for (a) are linearly dependent, while the others are independent since they are not identically zero.

3. A tank contains 180 liters of fluid in which 20 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 3 L/min; the well-mixed solution is pumped out at a rate of 4L/min. Find the number $y(t)$ of grams of salt in the tank at time t.

Solution: We the formula for the volume is $180 + (3 - 4)t = 180 - t$. Thus the concentration of salt at time t is $\frac{y(t)}{180-t}$. We thus have

$$\begin{aligned} \frac{dy}{dt} &= \text{natural increase rate} - \text{rate of decrease} \\ &= 3 - 4\frac{y(t)}{180 - t} \end{aligned}$$

This is a first order linear differential equation. Solving gives

$$\begin{aligned} y(t) &= (t - 180)^4((180 - t)^{-3} + C) \\ &= 180 - t + C(t - 180)^4. \end{aligned}$$

Using the given initial condition $y(0) = 20$ we see that

$$\begin{aligned} 20 &= 180 + C(180)^4 \\ \implies C &= \frac{-160}{180^4} \end{aligned}$$

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4. The function $y_1(x) = x^2\cos(\ln x)$ is a solution of the given differential equation

$$x^2y'' - 3xy' + 5y = 0.$$

Use reduction of order or the formula

$$y_2(x) = y_1(x)\int \frac{e^{\int -P(x)dx}}{y_1^2(x)}\,dx$$

to find a second solution $y_2(x)$ that is linearly independent with $y_1(x)$.

Solution: First obtain standard form of the second order differential equation:

$$y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0,$$

hence $P(x) = \frac{-3}{x}$. Now use the formula:

$$\begin{aligned}
y_2(x) &= y_1(x)\int \frac{e^{\int -P(x)dx}}{y_1^2(x)}dx \\
&= x^2\cos(\ln x)\int \frac{e^{\int \frac{3}{x}dx}}{(x^2\cos(\ln x))^2}dx \\
&= x^2\cos(\ln x)\int \frac{x^3}{(x^2\cos(\ln x))^2}dx \\
&= x^2\cos(\ln x)\int \frac{1}{x\cos^2(\ln x)}dx \\
&= x^2\cos(\ln x)\int \frac{1}{\cos^2(u)}du \\
&= x^2\cos(\ln x)(\tan(\ln x)) \\
&= x^2\sin(\ln x)
\end{aligned}$$

5. Suppose a population of rabbits on a meadow initially has the size $10,000$ and natural increase of 10% of the population per year. A pack of 20 wolves just migrated in recently and as a result, each wolf hunts for two rabbits each month for food. Suppose the population of wolves grows by 2 wolves/year. Set up a differential equation describing the population of rabbits over time. Will the population of rabbits eventually flourish, stay the same, or die out?

Solution: Let $y(t)$ denote the population of rabbits after t years, then

$$\frac{dy}{dt} = \text{natural increase rate} - \text{rate of decrease}$$

In this case, the decreasing rate of rabbits are determined by the hunting rate of the wolves. After t years, the wolf population is $20 + 2t$, so the rabbit population additionally decreases by a rate of $2 \times 12 \times (20 + 2t) = 48(t + 10)$. Hence, we have

$$\frac{dy}{dt} = 0.1y - 48(t + 10).$$

This is just a 1st order linear DE, solving which we get

$$y(t) = 48(10t + 200) + Ce^{0.1t}.$$

With the initial condition $y(0) = 10,000$, we have $C = 400$. Hence

$$\boxed{y(t) = 480t + 9600 + 400e^{0.1t}}.$$

Since $y'(t) = 480 + 40e^{0.1t} > 0$, the population of rabbits will keep increasing so it will eventually flourish.