Name: Date: 12/7/2023

M20580 L.A. and D.E.
Worksheet 13

1. 1. Let $\mathcal{B}$ be the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C}$ be the basis $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$. Find the change of basis matrix $P_{C \leftarrow B}$.

   (a) $\begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

   (e) none of these

Solution: $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ gives us the answer a.

2. Let M be the matrix

$$M = \begin{bmatrix} -2 & 1 & -2 & 2 \\ 17 & 32 & 20 & -18 \\ -16 & -24 & -15 & 16 \\ -7 & 2 & -3 & 6 \end{bmatrix}$$

   You are told that M has eigenvector

$$\begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

   What is the corresponding eigenvalue?

   (a) 2

   (b) 4

   (c) 5

   (d) 8

   (e) none of these

Solution: We only need to compute the first entry which is

$$-2 \cdot 2 + 1 \cdot 2 + (-2) \cdot 0 + 2 \cdot 5 = 8 = 4 \cdot 2$$

Thus, the answer is b.

3. Let $\mathcal{M}_{6\times 8}$ be the vector space of all 6 by 8 matrices, under addition of matrices and scalar multiplication of matrices. What is the dimension of $\mathcal{M}_{6\times 8}$?

   (a) 6
   (b) 8
   (c) 14
   (d) 48
   (e) none of these

Solution: The answer is d.

4. Let $T : \mathbb{R}^{18} \to \mathbb{R}^{14}$ be onto. What is the dimension of the kernel of T?

   (a) 4
   (b) 14
   (c) 0
   (d) 32
   (e) not enough information to tell

Solution: By rank-nullity theorem, $18 - 14 = 4$. Thus, the answer is a.

5. Which of the following is a subspace of the vector space of functions on the real numbers?

   (a) the set of f with $f(0) = 2$
   (b) the set of solutions to the differential equation $y'' - \sin(t)y = 0$
   (c) the set of polynomials of the form $at^3 + bt$ with $a \neq 0$
   (d) the set of solutions to the differential equation $y'' = 5$
   (e) none of these

Solution: The answer is b. For a, c and d, the zero function is not in the set so that the set cannot form a vector space of functions.

6. What is the dimension of the null space of $A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ -3 & -5 & -1 & -12 \\ 2 & 3 & 1 & 8 \\ 3 & 8 & -2 & 13 \end{bmatrix}$?

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) none of these

Solution: Turn matrix A into REF

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We will find that there is a zero row at the botom of the matrix. Thus, the dimension of the null space is 1.

7. Suppose a 7×7 matrix A has determinant 17. Which of the following must be TRUE?

   (a) the rank of A is 7
   (b) $\det(A^T) = \frac{1}{17}$
   (c) $\det(A^{-1}) = -17$
   (d) $\det(A^T A) = 49$
   (e) none of these

Solution: $\det(A) \neq 0$ which means the matrix is of full rank. Hence, a is correct.

8. Consider the initial value problem $y'' - 4y' - 5y = 0, y(0) = 1, y'(0) = 0$. Which of the following describes the behavior of the solution at $t \to +\infty$:

   (a) $\lim_{t\to+\infty} y(t) = +\infty$

(b) $\lim_{t\to+\infty} y(t) = -\infty$

(c) $\lim_{t\to+\infty} y(t) = 0$

(d) $y(t)$ is a decaying oscillation

(e) $y(t)$ is a growing oscillation

Solution: Using characteristic equation method, the solution to the ODE is

$$y = e^{1/6t} + e^{5/6t}.$$

y goes to ∞ as t goes to $infty$.

9. On which interval is the solution of the initial value problem $(\sin t)y'' + y = 1, y(1) = 1, y'(1) = 2$ certain to exist?

(a) $0 < t < 2\pi$

(b) $0 < t < \pi$

(c) $\pi < t < 2\pi$

(d) $-\infty < t < +\infty$

(e) cannot guarantee existence on any interval

Solution: We need to avoid t such that $\sin t = 0$ but also need to include $t = 1$, thus, b.

10. Consider the autonomous equation $y' = y(y-1)(y-2)(y-3)$ with initial condition $y(0) = 2.99$. Without solving the equation explicitly, find the limit $\lim_{t\to+\infty} y(t)$.

(a) 0

(b) 1

(c) 2

(d) 3

(e) ∞

Solution: c.