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Math 20580 Tutorial – Worksheet 2

1. Determine whether the vector $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v_2}$ and $\mathbf{v}_3$. If yes, find scalars a_1, a_2, a_3 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}.$$

Solution: To solve $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$, row reduce to the corresponding augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -3 & 1 & -6 & -5 \\ 0 & 1 & 8 & 6 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 9 & 1 \\ 0 & 1 & 8 & 6 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & -1 & 5 \end{array}\right]$$

By the third row, $a_3 = -5$. Similarly, we obtain $a_2 + 9a_3 = 1$ and $a_1 + 5a_3 = 2$. Thus $(a_1, a_2, a_3) = (27, 46, -5)$.

2. Find the inverses of the following matrices if it exists

$$(a) = \begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}, \qquad (b) = \begin{bmatrix} 4 & -6 \\ 6 & -9 \end{bmatrix}.$$

Solution: We recall that for a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if $ad - bc \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A does not have an inverse (not invertible).

(a) $2 \times 4 - 7 \times 3 = -13 \neq 0$, so the inverse is

$$\frac{1}{-13} \begin{bmatrix} 4 & -3 \\ -7 & 2 \end{bmatrix}.$$

(b)$4 \times (-9) - 6 \times (-6) = 0$, so the matrix is not invertible.

3. (a) Let

$$A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}.$$

Construct a 2×2 matrix B such that AB is the zero matrix. Use two different *nonzero* columns for B.

Solution: We note that if we write $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$, where $\mathbf{b}_1, \mathbf{b}_2$ are two column vectors in $\mathbb{R}^2$, then $AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2]$(See, e.g., textbook Theorem 3.3 and its proof). Hence, AB is a zero matrix if and only if $A\mathbf{b}_1 = A\mathbf{b}_2 = \mathbf{0}$, so we need to find two nonzero solutions to the system $A\mathbf{x} = \mathbf{0}$. Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The RREF of A is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

So we get $x_1 = 3x_2$ and any pair of x_1, x_2 satisfying this equation works. For example, We can choose $x_1 = 3, x_2 = 1$ or $x_1 = -3, x_2 = -1$, which gives

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix},$$

so one choice for a matrix B is

$$B = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}.$$

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Find the conditions on a, b, c and d such that A commutes with both B and C, that is, $AB = BA$ and $AC = CA$.

Solution: We can work out to see that

$$AB = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}, \qquad BA = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix},$$

so by comparing entry-by entry, we see that $AB = BA$ if and only if $b = 0 = c$. Likewise, we have

$$AC = \begin{bmatrix} a & -2a + b \\ c & -2c + d \end{bmatrix}, \qquad CA = \begin{bmatrix} a - 2c & b - 2d \\ c & d \end{bmatrix},$$

so that $AC = CA$ if and only iff $a = a - 2c$ and $-2a + b = b - 2d$ and $-2c + d = d$. Solving these equations, we get $c = 0$ and $a = d$.

4. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, T(\mathbf{w}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$ where $\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} + \mathbf{w}$.

Solution: We recall two properties of a linear transformation T: for any two vectors of appropriate size $\mathbf{u}, \mathbf{u}$ and any scalar c, we have

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

Now, we continually use these two properties to compute $T(\mathbf{x})$. We have

$$T(\mathbf{x}) = T(2\mathbf{u} + 3\mathbf{v} + \mathbf{w}) = 2T(\mathbf{u}) + 3T(\mathbf{v}) + T(\mathbf{u}).$$

So

$$T(\mathbf{x}) = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

(b) Continuing part (a), if we know

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^3$.

Solution: If we write

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for any vector in $\mathbb{R}^3$, we have

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

So using the same argument as in (b), we have

$$T(\mathbf{x}) = \begin{bmatrix} a + 3b + 4c \\ 2a + b + 2c \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

5. For each of the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the standard matrix for T, i.e., find a 2×2 matrix A such that $T\mathbf{x} = A\mathbf{x}$.

   (a)
   $$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \forall \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2.$$

   (b)
   $$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 3x_2 \end{bmatrix}, \forall \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2.$$

Solution:

(a)
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b)
$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}.$$