Name: Date: 09/07/2023

Math 20580 Tutorial
Worksheet 3

1. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

   (a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

   (b) A homogeneous linear system in $n \geq 1$ unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

   (c) If a homogeneous linear system of $n \geq 1$ equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

   (d) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

   (e) If a consistent linear system has more unknowns than equations, then it must have infinitely many solutions.

Solution:

(a) True. If a matrix is in reduced row echelon form, then in particular it is in row echelon form.

(b) True. We have an augmented matrix with r pivot columns. Therefore, there are $n - r$ columns which are not pivots, hence $n - r$ free variables.

(c) True. There are n pivot columns and zero free variables. So, there is a unique solution which must be the trivial one.

(d) False. Consider the augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. This matrix is in reduced row echelon form, but it has just one solution, the trivial one.

(e) True. If there are n unknowns and m equations where $n > m$, then there will be at most m pivot columns and therefore at least $n - m > 0$ free variables. So, there will be infinitely many solutions.

2. (graded)

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$. Compute $A^T A + 6B^{-1}$.

Solution:

$$\begin{aligned} A^T A + 6B^{-1} &= \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} + \frac{6}{2} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 14 \\ 14 & 21 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 26 \\ 14 & 24 \end{bmatrix} \end{aligned}$$

Correct A^T and B^{-1}, 1pt. Correct matrix multiplication $A^T A$, 1pt. Correct answer, 1pt.

(b) If the matrix A, B are such that

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$

then what is the matrix B?

Solution:

$$\begin{aligned} B &= A^{-1} AB \\ &= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

Correct A^{-1}, 1pt. Correct answer, 1pt.

3. Use Guass-Jordan method to find the inverse of the given matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 0 & 1 & 3 & -1 & -2 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 4 & -1 & -4 \\ -3 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix}$$

4. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(a) How many vectors are in the set $\{\vec{v}_1, \vec{v}_2\}$?

Solution: There are exactly 2 vectors in the set.

(b) How many vectors are in the set $\operatorname{span}(\vec{v}_1, \vec{v}_2)$?

Solution: There are infinitely many. Just consider $\vec{v}_1, 2\vec{v}_1, 3\vec{v}_1, 4\vec{v}_1, \ldots$. This is clearly an infinite set of vectors that lies in the given span.

(c) If $\vec{y} \in \operatorname{span}(\vec{v}_1)$, then must $\vec{y} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$? Please justify your answer.

Solution: Yes. Let $\vec{y} \in \operatorname{span}(\vec{v}_1)$, then $\vec{y} = c_1\vec{v_1} = c_1\vec{v}_1 + 0\vec{v}_2 \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$

(d) Prove that all vectors from $\mathbb{R}^3$ lie in $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. (Hint: row reduce the matrix whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and look at its rank)

Solution: row reduce the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, since this square matrix has rank 3, all vectors lie in $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

5. Given a matrix A, $Null(A)$ is the collection of all vectors v such that $Av = \mathbf{0}$, $Row(A)$ is the collection of all possible linear combinations of rows of A, and $Col(A)$ is the collection of all possible linear combinations of columns of A. Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Find a basis $\mathcal{B}$ for $Row(A)$.

(b) Find a basis $\mathcal{B}$ for $Col(A)$.

(c) Is $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ in $Null(A)$?

(d) Is $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix}$ in $Row(A)$?

Solution:

(a) Basis for $Row(A)$ is the collection of nonzero rows of REF of A.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 = 4R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 10 & -1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 = R_2 - 5R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 10 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

Thus, $\mathcal{B} = \{\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 10 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -6 \end{bmatrix}\}$

(b) Basis for $Col(A)$ is the collection of pivotal columns of A.

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) No. Because $A \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 2 \end{bmatrix}$ is not a zero matrix.

(d) Yes. Suppose $xR_1 + yR_2 + zR_3 = \begin{bmatrix} 6 & 4 & 2 \end{bmatrix}$, we get a linear system. We need to determine whether the linear system is consistent or not.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 2 & 10 & 0 & 4 \\ 0 & -1 & -6 & 2 \end{array}\right] \xrightarrow{R_2 = 2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & -10 & 0 & 8 \\ 0 & -1 & -6 & 2 \end{array}\right] \xrightarrow{R_3 = R_2 + 10R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & -10 & 0 & 8 \\ 0 & 0 & -60 & 28 \end{array}\right]$$

that implies the linear system is consistent.