Name: Date: 09/14/2023

M20580 L.A. and D.E. Tutorial
Worksheet 4

1. Let $\mathbf{b}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; and $\mathbf{c}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be two bases for $\mathbb{R}^2$. Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution: One method to solve is by putting the matrix $\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right]$ in reduced row echelon form. Another solution is to recall that $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} P_{\mathcal{E}\leftarrow\mathcal{B}}$ where $\mathcal{E}$ is the standard basis. We know that $P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$ and $P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. Calculate $P_{\mathcal{C}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{C}}^{-1}$ as $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$. Bringing it all together gives $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$.

2. Consider the following set of vectors $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \right\}$. If possible write the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of the given vectors. Does the given set span $\mathbb{R}^3$? What is the dimension of the span of these vectors?

Solution:

A linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{v}$ gives to a system of linear equations in the variables a_1, a_2 and a_3. Let us solve for this system:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & -3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right].$$

We see that the system is inconsistent and thus there is a no way to write $\mathbf{v}$ as a linear combination of the given vectors.

This tells us that the given set does not span $\mathbb{R}^3$ since $\mathbf{v}$ is a vector in $\mathbb{R}^3$ but $\mathbf{v}$ is not in the span.

Let us check if the vectors are linearly dependent.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This tells us that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0$ where a_3 is free, $a_2 = -a_3$, and $a_1 = 2a_3$. So for example $2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = 0$. So we can write $\mathbf{v}_3$ as $\mathbf{v}_2 - 2\mathbf{v}_1$, and so the span of the original three vectors is the same as the span of $\{\mathbf{v}_1\mathbf{v}_2\}$. These two are linearly independent, and thus they form a basis for $\text{Span}(\{\mathbf{v}_1\mathbf{v}_2\})$. Thus the dimension is 2.

(An alternative approach would be arguing that the span of the 3 vectors is the column space of the matrix $\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ and remembering that the dimension of the column space is the same as the rank of the matrix, and this matrix has rank 2.)

3. (a) Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for a vector space V. Also suppose that the change-of-basis matrix **from** $\mathcal{B}$ **to** $\mathcal{C}$ is given as

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

For $\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2$, what is $[\mathbf{v}]_{\mathcal{C}}$, the $\mathcal{C}$-coordinates for $\mathbf{v}$?

Solution: This question is graded for correctness. 1 point.

$\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2$ means that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So

$$[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\ [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

i.e. $[\mathbf{v}]_{\mathcal{C}} = 5\mathbf{c}_1 + 8\mathbf{c}_2$.

(b) Find the standard coordinates for $\mathcal{C}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$.

Solution: This part is worth 3 points. 1 point for using a suitable method, and 1 point each for $\mathbf{c}_1$ and $\mathbf{c}_2$.

We compute $P_{\mathcal{B}\leftarrow\mathcal{C}} = P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. So

$$[\mathbf{c}_1]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{c}_1]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $\mathbf{c}_1 = 2\mathbf{b}_1 - 3\mathbf{b}_3 = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$.

Similarly

$$[\mathbf{c}_2]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{c}_2]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So $\mathbf{c}_1 = -\mathbf{b}_1 + 2\mathbf{b}_3 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

An alternative approach would be to use the fact that $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow E}P_{\mathcal{E}\leftarrow B}$ and solve for $P_{\mathcal{E}\leftarrow\mathcal{C}}$.

(c) Calculate the standard coordinates for $\mathbf{v}$ from (a) using the standard coordinates for $\mathcal{B}$ given in (b) and also using the standard coordinates for $\mathcal{C}$ using your answers for (a) and (b) and check that they agree.

Solution: 1 point for the correct answers.

Using $\mathcal{B}$:

$$2\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}.$$

Using $\mathcal{C}$:

$$5\begin{bmatrix}-1\\-9\end{bmatrix}+8\begin{bmatrix}1\\7\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}.$$

4. Consider the basis $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}3\\4\\3\end{bmatrix}.\begin{bmatrix}1\\3\\0\end{bmatrix}\right\}$ for $\mathbb{R}^3$.

(a) If $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}2\\-1\\4\end{bmatrix}$, find $\mathbf{x}$ (its coordinate representation in the standard basis).

Solution:
$[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}2\\-1\\4\end{bmatrix}$ means that the coordinates of $\mathbf{x}$ relative to the $\mathcal{B}$ basis is $\begin{bmatrix}2\\-1\\4\end{bmatrix}$, so

$$\mathbf{x}=2\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}+(-1)\cdot\begin{bmatrix}3\\4\\3\end{bmatrix}+4\cdot\begin{bmatrix}1\\3\\0\end{bmatrix}=\begin{bmatrix}3\\10\\-1\end{bmatrix}.$$

(b) What is $P_{\mathcal{B}\leftarrow\mathcal{E}}$ where $\mathcal{E}$ is the standard basis?

Solution: We know that $P_{\mathcal{E}\leftarrow\mathcal{B}}=\begin{bmatrix}1&3&1\\1&4&3\\1&3&0\end{bmatrix}$ and that $P_{\mathcal{B}\leftarrow\mathcal{E}}=P_{\mathcal{E}\leftarrow\mathcal{B}}^{-1}$. Computing (using Gauss-Jordan elimination) gives $\begin{bmatrix}9&-3&-5\\-3&1&2\\1&0&-1\end{bmatrix}$.