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Math 20580 Linear algebra and differential equations tutorial
Worksheet 5

1. All of the following sets with their operations are **not** vector spaces. State at least one axiom of vector spaces that does **not** hold in each case and justify your answer with concrete examples:

   (a) The set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ with usual vector addition and scalar multiplication.

   (b) $\mathbb{R}^2$, with the usual scalar multiplication but addition is defined by

   $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix}.$$

   (c) The set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad = 0$ with usual matrix operations.

Solution:

(a) We have $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector of the proposed form, but $k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$ is not in the considered set if $k < 0$, so this set is not closed under scalar multiplication.

(b) Consider vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ with the scalar 2 we have

$$2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = 2\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

while

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \end{bmatrix}$$

so distributivity does not hold.

(c) The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in the considered set, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not, so this set is not closed under vector addition.

2. Determine whether the statements are true or false and justify your answer.

   (a) If $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all vectors v_1 and v_2 in V and all scalars c_1 and c_2 , then T is a linear transformation.

   (b) There is exactly one linear transformation $T : V \to W$ for which $T(u+v) = T(u-v)$ for all vectors u and v in V.

   (c) Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V. If $T(v_1) = T(v_2) = \cdots = T(v_n) = 0$, then T is the zero transformation.

   (d) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and if $[T]_{B\to B} = I_n$ with respect to some basis B of $\mathbb{R}^n$, then T is the identity transformation.

   (e) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and if $[T]_{B\to C} = I_n$ with respect to two different bases B and C of $\mathbb{R}^n$, then T is the identity transformation.

Solution:

(a) True. This follows from the definition of linear transformation. Since $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv) = cT(v)$, we have $T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

(b) True. If $T(u+v) = T(u-v)$, then $T(u)+T(v) = T(u)-T(v)$. So, $T(v) = -T(v)$ hence $2T(v) = 0$. So, $T(v) = 0$ for all vectors $v \in V$. Therefore, T is the zero transformation.

(c) True. In general, a linear transformation is determined by where it sends the basis vectors. Any vector $v \in V$ can be written as $c_1v_1 + c_2v_2 + \cdots + c_nv_n$. Applying T and using the definition of linear transformation, we have $T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0$ since each each $T(v_i) = 0$. So, T is the zero transformation.

(d) True. If $[T]_{B\to B} = I_n$ then the transformation does nothing to each coordinate vector relative to the basis B. In other words, letting $B = \{v_1, \ldots, v_n\}$, we have $T(v_1) = v_1, T(v_2) = v_2, \ldots, T(v_n) = v_n$. Any vector $v \in \mathbb{R}^n$ can be written as $v = c_1v_1 + \cdots + c_nv_n$, and we have $T(v) = T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n) = c_1v_1 + \cdots + c_nv_n = v$. So, T is the identity transformation.

(e) False. Let $n = 1$ and choose the bases $B = \{1\}$(i.e., standard basis) and $C = \{2\}$ for $\mathbb{R}$. The transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $T(1) = 2$ is not the identity, but the matrix relative to $\{1\}$ and $\{2\}$ is $[T]_{B\to C} = [1]$ since $T(1) = 2 = 1 \cdot 2$.

3. In each of the following, V is a vector space and W is a subset of V. Determine if W is a subspace of V. Justify your answer.

   (a) $V = \mathbb{R}^3$, and $W = \left\{ \begin{bmatrix} a \\ 0 \\ -2a \end{bmatrix} \mid a \in \mathbb{R} \right\}$,

   (b) $V = \mathcal{P}_3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$, and $W = \{a_1 + b_1 x + c_1 x^2 + d_1 x^3 \mid a_1, b_1, c_1, d_1 \in \mathbb{R} \text{ and } a_1 > b_1\}$,

   (c) $V = \mathcal{M}_{2\times 2}(\mathbb{R})$, and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } ad \geq 0 \right\}$.

Solution:

(a) $W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$ which is a subspace of $V = \mathbb{R}^3$. (See, e.g., Theorem 6.3 Poole's book).

(b) The polynomial $p(x) = 2 + x$ is in W, but $(-1)p(x) = -2 - x$ has $-2 < -1$ is not in W. Hence W is not closed under scalar multiplication, and is not a subspace of V.

(c) The matrices, $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ are in W, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is not. Since W is not closed under vector addition, W is not a subspace of V.

4. Recall that for any two polynomials $p = p(x), q = q(x)$ and any scalar λ, we define $p + q$ to be the polynomial that satisfies

$$(p+q)(x) = p(x) + q(x)$$

and the polynomial λp to be

$$(\lambda p)(x) = \lambda \cdot p(x).$$

Define a transformation $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find the kernel of T and describe the range of T.

Solution:

(a) Let $p = p(x)$ and $q = q(x)$ be two elements in $\mathcal{P}_2$. We have

$$T(p) + T(q) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix}$$

and

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix},$$

so $T(p+q) = T(p) + T(q)$. Moreover, for any scalar $\lambda \in \mathbb{R}$,

$$T(\lambda p) = \begin{bmatrix} (\lambda p)(0) \\ (\lambda p)(1) \end{bmatrix} = \begin{bmatrix} \lambda \cdot p(0) \\ \lambda \cdot p(1) \end{bmatrix} = \lambda \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \lambda T(p)$$

so T satisfies is a linear transformation.

(b) Writing $p(x) = a + bx + cx^2$, we have $T(p)$ is the zero vector if and only if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} a \\ a + b + c \end{bmatrix}.$$

This gives $a = a + b + c = 0$, which gives $b = -c$ and then $(a, b, c) = (0, t, -t)$, $t \in \mathbb{R}$.
Hence, $ker(T) = \{tx - tx^2 \mid t \in \mathbb{R}\}$.

Now, if $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2$ is in the range of T, then for some $p(x) = a + bx + cx^2$, we have

$$\begin{bmatrix} m \\ n \end{bmatrix} = T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

So $a = m$ and $n = a + b + c = m + b + c$. Letting $b = -m$ and $c = n$, we have $T(m - mx + nx^2) = \begin{bmatrix} m \\ n \end{bmatrix}$. Hence the range of T is the whole $\mathbb{R}^2$

5. $V = \mathcal{P}_3$ is a vector space, and $S = \{x^3 - 2x^2 + 3x - 1, 2x^3 + x^2 + 3x - 2\}$ is a subset of V. Determine if the given vector $v = -2x^3 - 11x^2 + 3x + 2$ of V is in span(S).

   If it is, find an explicit representation of v as a linear combination of the vectors in S.

Solution: The equation we are solving is

$$\begin{aligned} -2x^3 - 11x^2 + 3x + 2 &= a(x^3 - 2x^2 + 3x - 1) + b(2x^3 + x^2 + 3x - 2) \\ &= (a + 2b)x^3 + (-2a + b)x^2 + (3a + 3b)x + (-a - 2b) \end{aligned}$$

This is equivalent to the following system linear equations (by equating coefficients of same x-power term):

$$\begin{cases} a + 2b &= -2, \\ -2a + b &= -11, \\ 3a + 3b &= 3, \\ -a - 2b &= 2. \end{cases}$$

We have

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 5 & -15 \\ 0 & -3 & 9 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

The system has solution $a = 4$ and $b = -3$, so that

$$-2x^3 - 11x^2 + 3x + 2 = 4(x^3 - 2x^2 + 3x - 1) - 3(2x^3 + x^2 + 3x - 2).$$

Thus v is in span(S).