Name: Date: 09/28/2023

M20580 L.A. and D.E. Tutorial
Worksheet 6

1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ has out puts

$$T\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\2\\4\end{bmatrix}, \qquad T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\1\\3\end{bmatrix}.$$

Find $T\begin{bmatrix}4\\1\end{bmatrix}$.

Solution: The augmented matrix $\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 3 & 1 & 1 \end{array}\right]$ has REF$\left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \end{array}\right]$.Thus,

$$\begin{bmatrix}4\\1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix} + \frac{5}{2}\begin{bmatrix}2\\1\end{bmatrix}.$$

$$T\begin{bmatrix}4\\1\end{bmatrix} = -\frac{1}{2}T\begin{bmatrix}2\\3\end{bmatrix} + \frac{5}{2}T\begin{bmatrix}2\\1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1\\2\\4\end{bmatrix} + \frac{5}{2}\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}-1/2\\3/2\\11/2\end{bmatrix}.$$

2. Let $\mathcal{B} = \{1+t, 1-t^2, 1+t+t^2\}$ be a basis for the space of polynomials of degree at most 2. Find the coordinate vector $[p]_{\mathcal{B}}$ of $p(t) = 2+3t+4t^2$.

Solution: This question is the same as writing $p(t)$ as a linear combination of $1+t, 1-t^2, 1+t+t^2$. We pick standard basis for $\mathcal{P}_2$. Then we have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & -1 & 1 & 4 \end{array}\right], \text{ which has REF } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Hence, $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$.

3. Let $T : M_{2\times 2} \to \mathbb{R}^2$ be the linear transformation $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ d \end{bmatrix}$. Consider the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

   (a) Find the matrix of T relative to the standard basis for $M_{2\times 2}$ and the basis $\mathcal{B}$ of $\mathbb{R}^2$, i.e., find $[T]_{B\leftarrow std}$.

   (b) Find a basis for the kernel of T.

Solution:

(a) We have

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly,

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives $[T]_{std\leftarrow std} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The change of basis matrix from $\mathcal{B}$ to standard matrix in $\mathbb{R}^2$ is

$$[id]_{\mathcal{B}\leftarrow std} = [id]^{-1}_{std\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

So,

$$\begin{aligned} [T]_{\mathcal{B}\leftarrow std} &= [id \circ T]_{\mathcal{B}\leftarrow std} = [id]_{\mathcal{B}\leftarrow std}[T]_{std\leftarrow std} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

(b) Putting the matrix $[T]_{\mathcal{B}\leftarrow std}$ in RREF, we get $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Therefore, we have $a = -b - c$ and $d = 0$. So, a basis for the $\ker(T)$ is $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$.

4. (a) Let $S : \mathcal{P}_2 \to \mathcal{P}_3$ be the integration linear transformation, i.e.,

$$S(a_0 + a_1 x + a_2 x^2) = \int_0^x (a_0 + a_1 t + a_2 t^2)\ dt = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3.$$

Find the matrix representation of S with respect to the standard bases $\mathcal{A} = \{1, x, x^2\}$ for $\mathcal{P}_2$ and $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

(b) Find the matrix of the differentiation linear transformation $D : \mathcal{P}_3 \to \mathcal{P}_2$, i.e.,

$$D(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2$$

with respect to the standard bases for $\mathcal{P}_3$ and $\mathcal{P}_2$.

(c) What is the matrix representation of $D \circ S$ with respect to the standard basis $\mathcal{A}$ of $\mathcal{P}_3$, i.e., $\left[\frac{d}{dx}\int_0^x\right]$?

(d) Find a basis for the kernel of S and a basis for the kernel of D.

(e) Find a basis for the range of S and a basis for the range of D.

Solution:

(a) $[S]_{\mathcal{B}\leftarrow\mathcal{A}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$

(b) $[D]_{\mathcal{A}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(c) $[D]_{\mathcal{A}\leftarrow\mathcal{B}}[S]_{\mathcal{B}\leftarrow\mathcal{A}} = I_3.$

(d) There are no free variables in the matrix of S. This tells us immediately that the kernel of $[S]_{\mathcal{B}\leftarrow\mathcal{A}}$ is consists of just the zero vector, so the kernel of S is **the set contain only zero polynomial**. By convention, the basis for $\{\mathbf{0}\}$ is $\varnothing$ (the empty set).

In the matrix of D, we can see that there is one free variable (the first variable). Hence, a basis for the kernel of $[D]_{\mathcal{A}\leftarrow\mathcal{B}}$ is $\{(1, 0, 0, 0)^T\}$, so converting back to polynomial form, a basis for $\ker(D)$ is $\{1\}$ (the set of one polynomial given by the constant function 1).

(e) In order to obtain a basis for the range, we first find a basis for the column space of the matrix representation. For $[S]_{\mathcal{B}\leftarrow\mathcal{A}}$, we can take all three columns as a basis for $\text{col}([S]_{\mathcal{B}\leftarrow\mathcal{A}})$, so converting back to polynomial forms in $\mathcal{P}_3$ (because the range of S is a subset of $\mathcal{P}_3$), a basis for range(S) is $\{x, \frac{1}{2}x^2, \frac{1}{3}x^3\}$.

Similarly, we can take columns $2, 3, 4$ as a basis for $\text{col}([D]_{\mathcal{B}\leftarrow\mathcal{A}})$, and converting back to polynomial form gives us that a basis for range(D) is $\{1, 2x, 3x^2\}$.