Name: Date: 10/05/2023

M20580 L.A. and D.E. Tutorial
Worksheet 7

1. Let $T : \mathcal{P}_1 \to \mathcal{P}_1$ be the linear transformation defined by $T(p(x)) = p(x-1)$ and let $S : \mathcal{P}_1 \to \mathcal{P}_2$ be the linear transformation defined by $S(q(x)) = q(x^2 + x)$. Consider the bases $\mathcal{B} = \{1, x\}$ and $\mathcal{C} = \{1, x, x^2\}$ be the standard bases for $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Compute $[S \circ T]_{\mathcal{C} \leftarrow \mathcal{B}}$ in two ways:
   (a) By finding directly and then computing its matrix;
   (b) and by finding the matrices of S and T separately and using the fact that
   $$[S \circ T]_{\mathcal{C} \leftarrow \mathcal{B}} = [S]_{\mathcal{C} \leftarrow \mathcal{B}} [T]_{\mathcal{B}}.$$
   Hint: make sure your answers for (a) and (b) agree.

Solution: For (a) note that $S \circ T(1) = 1$ and $S \circ T(x) = S(x-1) = (x^2 + x) - 1$. Thus
$$[S \circ T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\ [S \circ T(1)]_{\mathcal{C}} \ \ : \ \ [S \circ T(x)]_{\mathcal{C}}\ \right] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
For (b) We have
$$[T]_{\mathcal{B}} = \left[\ [T(1)]_{\mathcal{B}} \ \ : \ \ [T(x)]_{\mathcal{B}}\ \right] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
and
$$[S]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\ [S(1)]_{\mathcal{C}} \ \ : \ \ [S(x)]_{\mathcal{C}}\ \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$
Multiplying $[S]_{\mathcal{C} \leftarrow \mathcal{B}}$ with $[T]_{\mathcal{B}}$ gives
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

2. Consider the linear transformation $T : \mathcal{P}_2 \to \mathbb{R}^3$ defined by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(-1) \end{bmatrix}.$$

Determine whether T is invertible as a linear transformation by considering its matrix with respect to the standard bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ respectively.

If it is invertible find it's inverse in terms of these standard bases, and also find a polynomial in $p(x) = a + bx + cx^2$ such that $T(p(x)) = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$.

Hint: recall that if T is invertible then $([T]_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}}$.

Solution: Recall that we have

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(1)]_{\mathcal{C}} \ : \ [T(x)]_{\mathcal{C}} \ : \ [T(x^2)]_{\mathcal{C}}\ \right].$$

2 points for Computing gives

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

1 point for Calculating the inverse gives

$$[T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ -1 & 1/2 & 1/2 \end{bmatrix}.$$

Multiplying this last matrix by $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$ gives

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

and thus $p(x) = 1 - x + 2x^2$ satisfies $T(p(x)) = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$. 2 points for calculating $p(x)$.

3. Let $\mathcal{E} := \{e^{2x}, e^{-2x}\}$ and let $M := \operatorname{span}\mathcal{E}$ be a space spanned by $\mathcal{E}$. Consider another basis $\mathcal{B} := \{\sinh(2x), \cosh(2x)\}$ of M, where

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}, \quad \cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}.$$

a) Find the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$.

b) Find the standard matrix $[D]_\mathcal{E}$ of the linear transformation $D : M \to M$ given by $D(f(x)) = \frac{df}{dx}$.

c) Find the standard matrix $[D]_\mathcal{B}$. **Hint:** recall that $[D]_\mathcal{B} = P_{\mathcal{B}\leftarrow\mathcal{E}}[D]_\mathcal{E} P_{\mathcal{E}\leftarrow\mathcal{B}}$.

Solution: Let us pick $\mathcal{E}$ as the standard basis of M. In terms of $\mathcal{E}$, the basis $\mathcal{B}$ is

$$\mathcal{B} = \{[\sinh(2x)]_\mathcal{E},\ [\cosh(2x)]_\mathcal{E}\} = \{\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix},\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\}.$$

Accordingly, the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$ is simply a concatenation of the latter columns:

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix};$$

its inverse is

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = 2\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Now, the operator D acts upon e^{2x} and e^{-2x} as

$$D(e^{2x}) = 2e^{2x}, \quad D(e^{-2x}) = -2e^{-2x},$$

hence

$$[D]_\mathcal{E} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Therefore,

$$[D]_\mathcal{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

A more direct way of computing $[D]_\mathcal{B}$ would be to notice that $D(\sinh(2x)) = 2\cosh(2x)$ and $D(\cosh(2x)) = 2\sinh(2x)$, which gives the columns of $[D]_\mathcal{B}$.

4. Let $T : M_{2\times 2} \to M_{2\times 2}$ be a linear transformation defined by

$$T(A) := AB - BA, \quad \text{where} \quad B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

a) Find the matrix $[T]_{\mathcal{E}}$ of T in the standard basis

$$\mathcal{E} = \left\{ E_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_3 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_4 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

b) Use $[T]_{\mathcal{E}}$ to compute the null space of T.

Solution: The action of T upon $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$T(A) = \begin{bmatrix} -a+b & a-b \\ -c+d & c-d \end{bmatrix} - \begin{bmatrix} -a+c & d-b \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}.$$

Therefore, the matrix of T in $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

Row reducing gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now solving for 0 gives two free variables, $c = s$ and $d = t$ and we have the solutions

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

i.e. the null space is Span $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$.