Name: Date: 10/12/2023

MATH20580 L.A. and D.E. Tutorial
Worksheet 8

1. Determine whether the statements are true or false. If false provide a counterexample. Let A and B be $n \times n$ matrices.

   (a) $\det(AB) = \det(A)\det(B)$

   (b) Let c be a real number. $\det(cA) = c^n \det(A)$

   (c) A is invertible if and only if $\det(A) \neq 0$

   (d) $\det(A + B) = \det(A) + \det(B)$

Solution:

(a) True.

(b) True.

(c) True.

(d) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 3 & 1 & -6 \end{bmatrix}$.

   (a) Compute the first column of the cofactor matrix associated to A.

   (b) Using part (a) compute the determinant of A using the Laplace expansion along the first column.

   (c) Check that you obtain the same answer using the usual formula for the determinant.

   (d) What can you conclude about the usual formula for the determinant and the Laplace expansion?

   (e) What is $\mathrm{adj}(A)A$? The adjoint or (adjugate) of A, $\mathrm{adj}(A)$ is the transpose of the cofactor matrix.

Solution:

(a) We compute C_{11} C_{21} and C_{31}.

$$C_{11} = (-1)^{1+1}\det\left(\begin{bmatrix} 0 & -3 \\ 1 & -6 \end{bmatrix}\right) = (-1)^{1+1}((0)(-6) - (-3)(1)) = 3.$$

$$C_{21} = (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -6 \end{bmatrix}\right) = (-1)^{2+1}((1)(-6) - (1)(1)) = 7.$$

$$C_{31} = (-1)^{3+1}\det\left(\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}\right) = (-1)^{3+1}((1)(-3) - (1)(0)) = -3.$$

(b) The Laplace expansion along the first column of A is $0C_{11} + 1C_{21} + 3C_{31} = 0(3) + 1(7) + 3(-3) = -2$.

(c) The usual formula (Laplace expansion across the top row) yields
$0((0)(-6) - (1)(3)) - 1((1)(-6) - (-3)(3)) + 1((1)(1) - (0)(3)) = -3 + 1 = -2$.

(d) We should see that the usual formula for the determinant is a special case of the Laplace expansion where we expand across the top row.

(e) We should get $\mathrm{adj}(A)A = \det(A)I$.

3. Use Cramer's rule to solve the system:

$$\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

Solution: According to Cramer's rule, to obtain x_1 we replace the first column of the matrix with the solution column and take the determinant of that divided by the determinant of the original matrix. Thus we have

$$x_1 = \frac{\det\left(\begin{bmatrix} 1 & -4 \\ 17 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix}\right)} = \frac{1-(-4)(17)}{3-(-4)(5)} = \frac{1+68}{3+20} = \frac{69}{23} = 3$$

Similarly, to get x_2 we replace the second column of the matrix with the solution vector, take the determinant and divide by the determinant of the original matrix: Thus we have

$$x_2 = \frac{\det\left(\begin{bmatrix} 3 & 1 \\ 5 & 17 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix}\right)} = \frac{(3)(17)-(5)(1)}{23} = \frac{51-5}{23} = \frac{46}{23} = 2.$$

Finally, we check that

$$\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

   (a) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

   (b) What is A^{2022}?

Solution:

(a) • First, we find all eigenvalues of A:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5),$$

which implies that the eigenvalues are $\lambda = -2$ and $\lambda = 5$. Since the eigenvalues of A are **distinct**, A is **diagonalizable**.

• Next, we find eigenvectors corresponding to each eigenvalue:

- For $\lambda = -2$:

$$(A + 2I_2)\mathbf{x} = 0 \Longleftrightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

and hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \ t \in \mathbb{R}.$$

- For $\lambda = 5$: Using the same method, the eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

• According to the theory of diagonalization, the following matrices

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

satisfies $A = PDP^{-1}$.

(b) Using that $A = PDP^{-1}$, we can simplify:

$$\begin{aligned} A^{2022} &= (PDP^{-1})^{2022} = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \qquad \text{(2022 times)} \\ &= PD(P^{-1}P)D(P^{-1}P)D\dots(P^{-1}P)DP^{-1} \\ &= PD^{2022}P^{-1} \end{aligned}$$

Let's compute P^{-1} and D^{2022}:

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \implies P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix};$$
$$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \implies D^{2022} = \begin{bmatrix} (-2)^{2022} & 0 \\ 0 & 5^{2022} \end{bmatrix}.$$

Finally,

$$\begin{aligned} A^{2022} &= PD^{2022}P^{-1} \\ &= \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} (-2)^{2022} & 0 \\ 0 & 5^{2022} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} (3)5^{2022} + (4)2^{2022} & 4 \cdot 5^{2022} - (4)2^{2022} \\ (3)5^{2022} - (3)2^{2022} & 3 \cdot 2^{2022} + (4)5^{2022} \end{bmatrix} \end{aligned}$$

Name: Date: 10/12/2023

5. Let $A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$.

(a) Determine all the eigenvalues of A.

(b) For each eigenvalue λ of A, find the eigenspace E_λ.

(c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(d) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution:

(a) We have

$$\det(A - \lambda I_3) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$.

(b) Recall that a λ-eigenvector is an element of the kernel of $A - \lambda I$.

We have that

$$A-\lambda_1 I = \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \quad A-\lambda_2 I = \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \quad A-\lambda_3 I = \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

The eigenspaces E_λ of A will be the null spaces $A - \lambda I$. We find these using row reduction on the homogeneous systems $[A - \lambda I|\vec{0}]$.

For $\lambda_1 = 1$, we have $E_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$, we have $E_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_3 = 3$, we have $E_3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(c) A basis for $\mathbb{R}^3$ consisting of eigenvectors of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(d) The matrices P and D we need to find are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$