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Math 20580 L.A. and D.E. Tutorial
Worksheet 9

1. Determine whether the statements are true or false. If true, quickly explain why, and give a counterexample if false.

   (a) Every linearly independent set of vectors is orthogonal.

   (b) Every orthogonal set of vectors is linearly independent.

   (c) Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

   (d) For any vector $\mathbf{x}$, the vector $\text{proj}_W\mathbf{x}$ is orthogonal to every vector in the vector space W.

   (e) Every orthogonal set is orthonormal.

   (f) Suppose A is an $n \times n$ diagonalizable matrix with eigenvalues $a_1, a_2, \ldots, a_n$ (not necessarily distinct), then the trace of A (the sum of diagonal entries) is $a_1 + a_2 + \cdots + a_n$

   (g) Suppose A is an $n \times n$ diagonalizable matrix with eigenvalues $a_1, a_2, \ldots, a_n$ (not necessarily distinct), then the determinant of A is $a_1 \cdot a_2 \cdot \ldots a_n$.

(Hint for (f) and (g): First, recall that for two square matrices A and B of the same size, we have $\text{trace}(AB) = \text{trace}(BA)$ and $\det(AB) = \det(A)\det(B)$. Now if A is similar to a diagonal matrix D, what are the relation between the trace or determinant of A and the trace or determinant D?)

Solution:

(a) False. In $\mathbb{R}^2$, the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ form a basis, hence they are linearly independent, but they are not orthogonal since $\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 1 \neq 0$.

(b) False. Any orthogonal set of **nonzero** vectors is linearly independent. The set $\{\mathbf{0}\}$ is orthogonal but not linearly independent.

(c) True. Any nontrivial subspace has a basis, and we can use the Gram–Schmidt algorithm to find an orthonormal basis.

(d) False. W contains the vector $\text{proj}_W\mathbf{x}$. What is true is that $\mathbf{x} - \text{proj}_W\mathbf{x}$ is orthogonal to every vector in W.

Take, for instance, $W = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ and $\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}$; then $\text{proj}_W\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ is not orthogonal to W.

(e) False. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$ is orthogonal but not orthonormal.

(f) True. Since A is diagonalizable, there exists an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix, with **diagonal entries that are exactly the eigenvalues** $a_1, \ldots, a_n$ of A (up to permutation of diagonal entries).

Hence $\text{trace}(D) = a_1 + a_2 + \cdots + a_n$ and A and D are **similar matrices**.

Using the hint with PD and P^{-1}, we have

$$\text{trace}(A) = \text{trace}(PDP^{-1}) = \text{trace}(P^{-1}PD) = \text{trace}(D) = a_1 + a_2 + \cdots + a_n.$$

(g) True. Similar to part (f), we have $\det(D) = a_1 a_2 \ldots a_n$. Since $A = PDP^{-1}$, we have

$$\begin{aligned}\det(A) &= \det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1}) \\ &= \det(D)[\det(P)\det(P^{-1})] \\ &= \det(D)\det(PP^{-1}) \\ &= \det(D)\det(I_n) = \det(D) = a_1 a_2 \ldots a_n.\end{aligned}$$

2. Let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ 2 \end{bmatrix} \right\}$ be a subspace of $\mathbb{R}^4$.

Find a basis for the orthogonal complement $V^\perp$ of V.

Solution: If $\mathbf{x}$ is in $V^\perp$, then $\mathbf{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} = 0$ and $\mathbf{x} \cdot \begin{bmatrix} 2 \\ 0 \\ 4 \\ 2 \end{bmatrix} = 0$.

This implies that $\mathbf{x}$ is in the null space of $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 4 & 2 \end{bmatrix}$, so we need to solve $A\mathbf{x} = \mathbf{0}$.

The augmented matrix $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 2 & 0 & 4 & 2 & 0 \end{array}\right]$ has the RREF $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right]$.

From this, we see that:

$$\mathbf{x} = \begin{bmatrix} -2s - t \\ 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, a basis for $V^\perp$ is span $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

3. It is known that the set $\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$ (you don't need to verify this).

   Use Gram–Schmidt process to find an orthonormal basis for $\mathbb{R}^4$ from $\mathcal{B}$.

Solution: For notational convenience, we will write column vector as a

First, we fix $\mathbf{u}_1 = \mathbf{v}_1$ and obtain the unit vector

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{(1,1,1,1)^T}{\sqrt{1^2+1^2+1^2+1^2}} = \frac{1}{2}(1,1,1,1)^T$$

.

Next, we have

$$\begin{aligned}
\mathbf{u}_2 &= \mathbf{v}_2 - \mathbf{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 \\
&= (-1,1,1,1)^T - \left[(-1,1,1,1)^T \cdot \frac{1}{2}(1,1,1,1)^T\right]\frac{1}{2}(1,1,1,1)^T \\
&= \frac{1}{2}(-3,1,1,1)^T.
\end{aligned}$$

Normalizing $\mathbf{u}_2$ gives

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{\frac{1}{2}(-3,1,1,1)^T}{\frac{1}{2}\sqrt{(-3)^2+1^2+1^2+1^2}} = \frac{1}{2\sqrt{3}}(-3,1,1,1)^T$$

.

For the third vector, we have

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{v}_3 - \mathbf{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \mathbf{proj}_{\mathbf{u}_2}\mathbf{v}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{v}_3 \cdot \mathbf{e}_2)\mathbf{e}_2 \\
&= (1,-1,1,1)^T - \left[(1,-1,1,1)^T \cdot \frac{1}{2}(1,1,1,1)^T\right]\frac{1}{2}(1,1,1,1)^T \\
&\qquad - \left[(1,-1,1,1)^T \cdot \frac{1}{2\sqrt{3}}(-3,1,1,1)^T\right]\frac{1}{2\sqrt{3}}(-3,1,1,1)^T \\
&= \frac{1}{3}(0,-4,2,2)^T
\end{aligned}$$

Normalizing $\mathbf{u}_3$ gives

$$\mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{\frac{1}{3}(0,-4,2,2)^T}{\frac{1}{3}\sqrt{0^2+(-4)^2+2^2+2^2}} = \frac{1}{2\sqrt{6}}(0,-4,2,2)^T$$

.

Finally, for the last vector we have

$$
\begin{aligned}
\mathbf{u}_4 &= \mathbf{v}_4 - (\mathbf{v}_4 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{v}_4 \cdot \mathbf{e}_2)\mathbf{e}_2 - (\mathbf{v}_4 \cdot \mathbf{e}_3)\mathbf{e}_3 \\
&= (1,1,-1,1)^T - \left[(1,1,-1,1)^T \cdot \frac{1}{2}(1,1,1,1)^T\right]\frac{1}{2}(1,1,1,1)^T \\
&\qquad\qquad - \left[(1,1,-1,1)^T \cdot \frac{1}{2\sqrt{3}}(-3,1,1,1)^T\right]\frac{1}{2\sqrt{3}}(-3,1,1,1)^T \\
&\qquad\qquad - \left[(1,1,-1,1)^T \cdot \frac{1}{2\sqrt{6}}(0,-4,2,2)^T\right]\frac{1}{2\sqrt{6}}(0,-4,2,2)^T \\
&= (0,0,-1,1)^T
\end{aligned}
$$

Normalizing $\mathbf{u}_4$ gives

$$
\mathbf{e}_4 = \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} = \frac{(0,0,-1,1)^T}{\sqrt{0^2+0^2+(-1)^2+1^2}} = \frac{1}{\sqrt{2}}(0,0,-1,1)^T
$$

.

Hence, an orthonormal basis for $\mathbb{R}^4$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ where

$$
\mathbf{e}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ \mathbf{e}_2 = \frac{1}{2\sqrt{3}}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},\ \mathbf{e}_3 = \frac{1}{2\sqrt{6}}\begin{bmatrix}0\\-4\\2\\2\end{bmatrix},\ \mathbf{e}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}.
$$

4. Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Solution: First we use Gram-Schmidt process to produce an orthogonal set.

$$v_1 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Then the orthonormal basis for col(A) is

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}.$$

$A = QR$ for some upper triangular matrix R, to find R we use the fact that Q has orthonormal columns, hence $Q^T Q = I$. Therefore $Q^T A = Q^T Q R = IR = R$

$$R = Q^T A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & -2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}.$$

$$A = QR = \begin{bmatrix} 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$