**Math 20580**
**Midterm 3**
**April 16, 2020**

Name:________________________
Instructor:________________________
Section:________________________

Calculators are NOT allowed. You will be allowed 120 minutes to do the test.

There are 15 multiple choice questions worth 6 points each and you will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

**Sign the pledge.** "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b ~~c~~ d e
2. a b ~~c~~ d e
3. ~~a~~ b c d e
4. a b c ~~d~~ e
5. a ~~b~~ c d e
6. a b c d ~~e~~
7. a b c ~~d~~ e
8. ~~a~~ b c d e
9. a ~~b~~ c d e
10. a b ~~c~~ d e
11. a ~~b~~ c d e
12. a b c d ~~e~~
13. a b ~~c~~ d e
14. a b c d ~~e~~
15. a b c ~~d~~ e

1. The differential equation

$$\frac{dy}{dt} + ty^2 = 0$$

is

(a) linear (b) a partial differential equation (c) separable
(d) an equation of order 2 (e) none of the above

$$\frac{dy}{dt} = -ty^2 \iff$$

$$\frac{dy}{y^2} = -t\,dt$$

separable

2. Let $A$ be an $m$ by $n$ matrix. Which of the following must be true:

(a) $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$.
(b) $A\vec{x} = \vec{b}$ always has a unique least-squares solution.
(c) $A^TA\vec{x} = A^T\vec{b}$ has a solution for any $\vec{b}$.
(d) $\vec{b}$ is orthogonal to $A\vec{x}$ for any $\vec{x}$.
(e) None of the above.

The normal equation always has a solution!

3. The matrix $\begin{bmatrix} -2 & 2 \\ -15 & 9 \end{bmatrix}$ has eigenvalues 3 and 4. Then we know that $A = PDP^{-1}$ for $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $P$ given by:

(a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 9 & -2 \\ 15 & -2 \end{bmatrix}$

$E_3 = \mathrm{Nul}\begin{bmatrix} -2-3 & 2 \\ -15 & 9-3 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$

$E_4 = \mathrm{Nul}\begin{bmatrix} -2-4 & 2 \\ -15 & 9-4 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

4. Suppose you wish to solve the differential equation $y' + \frac{2}{t}y = t^4$, $t > 0$, using integrating factors. After multiplying the equation by the integrating factor $\mu(t)$, the equation becomes:

(a) $ty' + 2y = t^5$ (b) $e^t ty' + 2e^t y = e^5 t^5$ (c) $e^{2t} ty' + 2e^{2t} y = e^{2t} t^5$
(d) $t^2 y' + 2ty = t^6$ (e) none of the above

$\mu(t) = e^{\int \frac{2}{t} dt}$

$= e^{2 \ln t}$

$= (e^{\ln t})^2 = t^2$

Get $t^2 y' + 2ty = t^6$

5. Applying the Gram-Schmidt process to the vectors $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ gives

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ (e) None of the above

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \frac{15}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

6. Find the general solution to $y' = \dfrac{y^2}{t^3}$, $t > 0$.

(a) $y = 2t^2 + C$ (b) $y = Ce^{2t} + 3$ (c) $y = -t^2 + C$ (d) $y = e^{3t} + C$

(e) None of the above

$\int \frac{dy}{y^2} = \int \frac{dt}{t^3}$

$-\frac{1}{y} = \frac{-1}{2t^2} + c$

$y = \frac{-1}{\frac{-1}{2t^2} + c} = \frac{2t^2}{1 - c \cdot 2t^2}$

7. The vector $\vec{v} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) $\lambda = 2i$ (b) $\lambda = 2+i$ (c) $\lambda = -2+3i$ (d) $\lambda = 2-i$ (e) $\lambda = -1-i$

$$\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -3-i \\ 2-i \end{bmatrix}$$

$$= \lambda \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

$\lambda = 2-i$

8. Recall that $\mathbb{P}_n$ denotes the vector space of polynomials of degree at most $n$, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$T(y) = y'' - y' + ty.$$

The matrix of $T$ relative to the basis $\{1, t, t^2\}$ of $\mathbb{P}_2$ and the basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

(a) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} t^2 \\ -2t \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} t^3 \\ -3t^2 \\ 6t \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

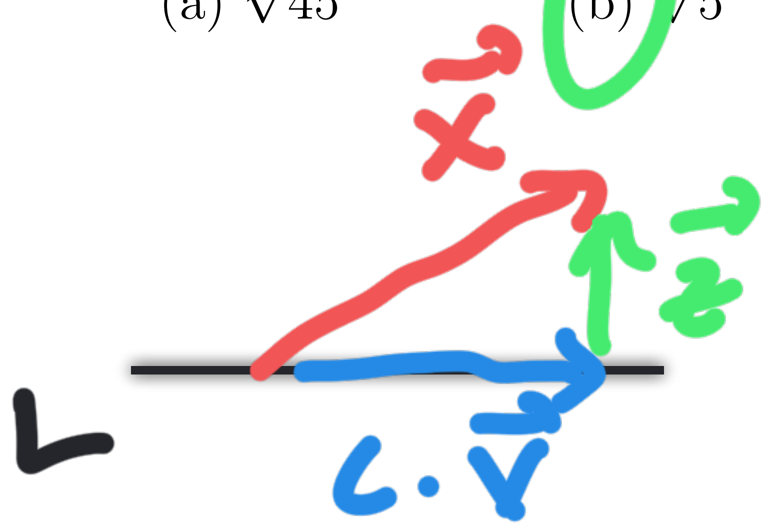
(e) it cannot be determined from the given information.

$T(1) = t$ $T(t) = -1+t^2$, $T(t^2) = 2-2t+t^3$

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Consider the line $L$ spanned by the vector $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The distance from the vector $\vec{x} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$ to the line $L$ is

(a) $\sqrt{45}$ (b) $\sqrt{5}$ (c) $2\sqrt{3}$ (d) 5 (e) $\sqrt{50}$




$$c = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{15}{5} = 3$$

$$\vec{z} = \vec{x} - 3\vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\|\vec{z}\| = \sqrt{1+4} = \sqrt{5}$$

10. Which of the following functions is a solution of the initial value problem

$$(y' - \sin x)^2 = 4 - 4y^2, \quad y(0) = 1$$

(a) $\cancel{-\sin x}$ (b) $\cancel{1}$ (c) $\cos x$ (d) $\cancel{x \cos x - \sin x}$ (e) $\cancel{\sin x - \cos x}$

$x = 0$: (a) 0 (d) 0 (e) $-1$

try: $y = 1$, $y' = 0$

$$(1 - \sin x)^2 \overset{?}{=} 4 - 4 = 0 \quad \text{No}$$

try: $y = \cos x$, $y' = -\sin x$

$$(-\sin x - \sin x)^2 \overset{?}{=} 4 - 4\cos^2 x$$

$$\Leftrightarrow \quad 4\sin^2 x = 4(1 - \cos^2 x) \quad \text{Yes}$$

11. Consider the matrices

free

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 & 11 \\ 3 & 6 & -3 & 1 & 15 \\ -1 & -2 & 1 & 2 & 2 \\ 4 & 8 & -4 & 4 & 28 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $B$ is the reduced echelon form of $A$. A basis for the orthogonal complement of the row space of $A$ is given by

(a) $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -3 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ -4 \\ 4 \\ 28 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 4 \\ 6 \\ -2 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 11 \\ 15 \\ 2 \\ 28 \end{bmatrix} \right\}$

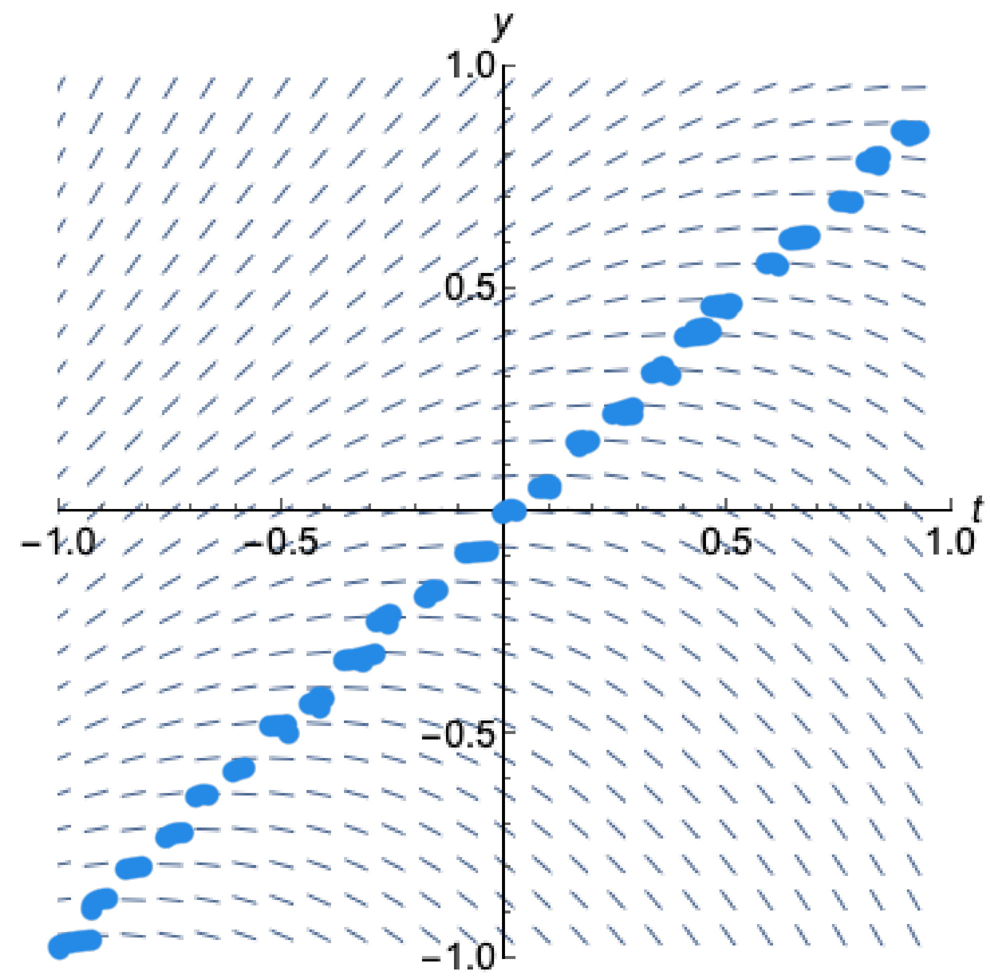
$$(\text{Row } A)^{\perp} = \text{Nul}(A)$$

$$\begin{aligned} x_1 &= -2s + t - 4u \\ x_2 &= s \\ x_3 &= t \\ x_4 &= -3u \\ x_5 &= u \end{aligned} \qquad \vec{x} = \begin{bmatrix} -2s+t-4u \\ s \\ t \\ -3u \\ u \end{bmatrix}$$

$$= s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

12. Determine $f(t, y)$ if the differential equation $y' = f(t, y)$ has direction field (the value of $t$ is measured on the horizontal axis, and the value of $y$ on the vertical axis)

As t increases slopes decrease




As y increases, slopes decrease

slope 0 along line $y = t$

(a) $t + y$ (b) $t - y$ (c) $y$ (d) $-t$ (e) $y - t$

13. Consider the line $L$ spanned by the vector $\vec{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$, and let $\text{proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation that sends a vector to its orthogonal projection onto the line $L$. The standard matrix of the transformation $\text{proj}_L$ is

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} 3 & 5 \\ 5 & -4 \end{bmatrix}$ (c) $\frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$ (d) $\frac{1}{5}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$

(e) none of the above

$U = [\vec{u}]$ ← orthonormal basis for $L$

Matrix is $UU^T =$

$$= \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$$

14. Find the solution to the initial value problem

$$\frac{dy}{dt} - ty = t, \qquad y(0) = 1.$$

(a) $y = e^{2t^2}$ (b) $t = \ln(1-y)$ (c) $y = e^t$
(d) $y = 2 - t^2$ (e) $y = 2e^{t^2/2} - 1$

$\mu(t) = e^{\int -t\,dt} = e^{-t^2/2}$

$y(t) = \dfrac{\int e^{-t^2/2} \cdot t\, dt}{e^{-t^2/2}} = \dfrac{-e^{-t^2/2} + C}{e^{-t^2/2}}$

$= -1 + C \cdot e^{t^2/2}$

$t=0,\ y=1 = -1 + C$
$\Rightarrow C = 2$

$y(t) = -1 + 2e^{t^2/2}$

15. The $QR$ factorization of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ -2 & 2 \end{bmatrix}$ has $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$. What is $R$?

(a) $\frac{1}{3}\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 3 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 3 \\ 0 & 6 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 1 \\ 0 & -3 \end{bmatrix}$.

$R = Q^T A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 4 \\ -2 & 2 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 9 & 9 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 6 \end{bmatrix}$