Math 20580
Final Exam
May 7, 2020

Name:________________________
Instructor:________________________
Section:________________________

Calculators are NOT allowed. You will be allowed 180 minutes to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1.	~~a~~	b	c	d	e	11.	~~a~~	b	c	d	e
2.	a	~~b~~	c	d	e	12.	a	b	c	d	~~e~~
3.	a	b	c	~~d~~	e	13.	a	b	~~c~~	d	e
4.	~~a~~	b	c	d	e	14.	a	b	c	~~d~~	e
5.	a	b	c	~~d~~	e	15.	a	~~b~~	c	d	e
6.	a	b	~~c~~	d	e	16.	a	b	~~c~~	d	e
7.	a	~~b~~	c	d	e	17.	~~a~~	b	c	d	e
8.	a	b	c	~~d~~	e	18.	a	b	c	d	~~e~~
9.	a	~~b~~	c	d	e	19.	a	~~b~~	c	d	e
10.	~~a~~	b	c	d	e	20.	a	b	c	~~d~~	e

1. Let $\mathcal{B}$ be the basis $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ and $\mathcal{C}$ be the basis $\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

(a) $\begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
(e) none of these

$$\left[\begin{array}{cc:cc} 2 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 2 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 1 & 1 \end{array}\right]$$

2. Let M be the matrix $\begin{bmatrix} -2 & 1 & -2 & 2 \\ 17 & 32 & 20 & -18 \\ -16 & -24 & -15 & 16 \\ -7 & 2 & -3 & 6 \end{bmatrix}$. You are told that M has eigenvector $\begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) 2 (b) 4 (c) 5 (d) 8 (e) none of these

$$M \cdot \begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -2\cdot 2 + 1\cdot 2 + 2\cdot 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 8 \\ \vdots \end{bmatrix}$$

$\lambda = 4$

3. Let $M_{6\times 8}$ be the vector space of all 6 by 8 matrices, under addition of matrices and scalar multiplication of matrices. What is the dimension of $M_{6\times 8}$?

(a) 6 (b) 8 (c) 14 (d) 48 (e) none of these

4. Let $T : \mathbb{R}^{18} \to \mathbb{R}^{14}$ be onto. What is the dimension of the kernel of T?

(a) 4 (b) 14 (c) 0 (d) 32 (e) not enough information to tell

$18 - 14 = 4$

(rank nullity theorem)

5. What is the general solution of the equation $y'' - 4y' + 5y = 0$?

(a) $c_1e^{-4t} + c_2e^{5t}$ (b) $c_1e^{4t} + c_2\cos(5t)$ (c) $c_1e^{4t} + c_2e^{t}$
(d) $c_1e^{2t}\cos(t) + c_2e^{2t}\sin(t)$ (e) $c_1e^{2t} + c_2e^{-3t}$

$r^2 - 4r + 5 = 0$

$(r-2)^2 + 1 = 0$

$r - 2 = \pm i$

$r = 2 \pm i$

FSS

$\{e^{2t}\cos t, e^{2t}\sin t\}$

6. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2, and let $\mathbb{B} = \{1, t+1, (t+1)^2\}$. With respect to $\mathbb{B}$, the coordinates of $3t^2 + 2t + 1$ are:

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (e) none of the above

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$c_1 = 2$

$c_2 = -4$

$c_3 = 3$

7. Which of the following is a subspace of the vector space of functions on the real numbers?

(a) the set of f with $f(0) = 2$
(b) the set of solutions to the differential equation $y'' - \sin(t)y = 0$
(c) the set of polynomials of the form $at^3 + bt$ with $a \neq 0$
(d) the set of solutions to the differential equation $y'' = 5$
(e) none of these

8. Which of the following sets of vectors are linearly independent?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 15 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ (e) none of these

not scalar multiples
of each other

9. What is the dimension of the null space of $A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ -3 & -5 & -1 & -12 \\ 2 & 3 & 1 & 8 \\ 3 & 8 & -2 & 13 \end{bmatrix}$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of these 2

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

10. Suppose a 7x7 matrix A has determinant 17. Which of the following must be TRUE?

(a) the rank of A is 7 (b) $\det A^T = \dfrac{1}{17}$ (c) $\det A^{-1} = -17$
(d) $\det(A^T A) = 49$ (e) none of these

$\det A = 17 \neq 0$

So invertible

$\rightarrow \text{rank } A = 7$

11. Consider the initial value problem $y'' - 4y' - 5y = 0$, $y(0) = 1$, $y'(0) = 0$. Which of the following describes the behavior of the solution at $t \to +\infty$:

(a) $\lim_{t\to\infty} y(t) = +\infty$ (b) $\lim_{t\to\infty} y(t) = -\infty$ (c) $\lim_{t\to\infty} y(t) = 0$
(d) $y(t)$ is a decaying oscillation (e) $y(t)$ is a growing oscillation

$r^2 - 4r - 5 = (r-2)^2 - 9 = (r-5)(r+1)$

$y = c_1 e^{5t} + c_2 e^{-t}$

$y' = 5c_1 e^{5t} - c_2 e^{-t}$, $\begin{cases} c_1 + c_2 = 1 \\ 5c_1 - c_2 = 0 \end{cases}$

$c_1 = \frac{1}{6}$, $c_2 = \frac{5}{6}$ $\Rightarrow \lim_{t\to\infty} y(t) = A$

12. Consider the equation $y'' + t^2y' + t^3y = 0$. Let y_1 be the solution satisfying $y_1(0) = 1$, $y_1'(0) = 2$ and let y_2 be the solution satisfying $y_2(0) = 3$, $y_2'(0) = 4$. Using Abel's formula, find the Wronskian $W[y_1, y_2]$.
(*Hint:* fix the constant in Abel's formula by computing $W[y_1, y_2]$ at $t = 0$ directly from the initial conditions on y_1, y_2.)

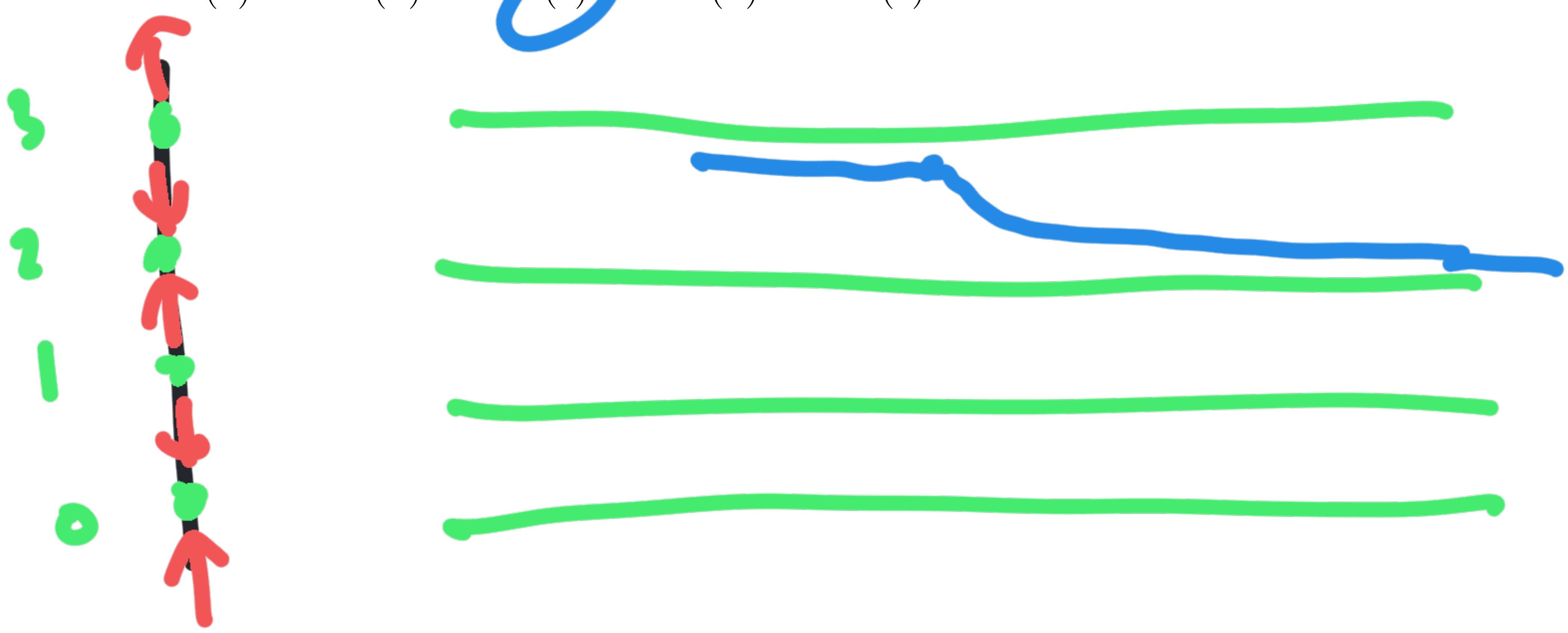
(a) 0 (b) $-2e^{-t^4/4}$ (c) e^t (d) $e^{t^3/3}$ (e) $-2e^{-t^3/3}$

$W = c \cdot e^{-\int t^2 dt} = c \cdot e^{-t^3/3}$

$W(0) = c$
$= \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = -2$

13. Consider the autonomous equation $y' = y(y-1)(y-2)(y-3)$ with initial condition $y(0) = 2.99$. Without solving the equation explicitly, find the limit $\lim_{t \to +\infty} y(t)$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) ∞



14. Equation $\overbrace{x^2y + \sin x}^{M} + (\overbrace{\frac{x^3}{3} + e^y}^{N})y' = 0$ is:

(a) linear (b) autonomous (c) separable (d) exact
(e) none of the above

$M_y = x^2$

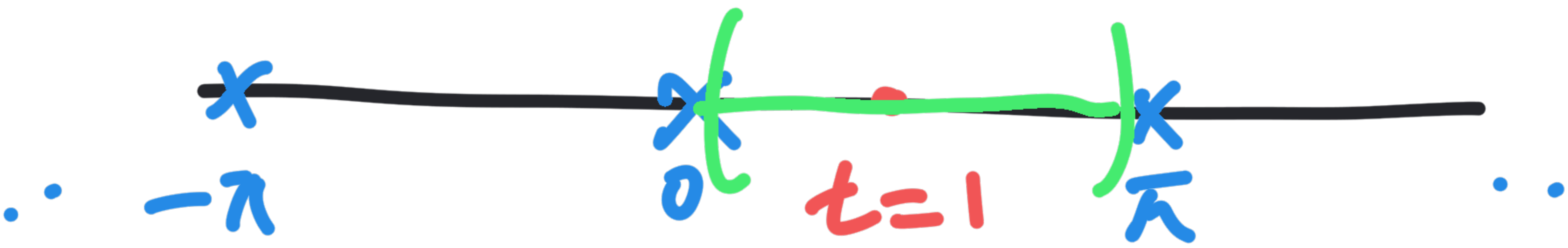
$N_x = x^2$

15. On which interval is the solution of the initial value problem $(\sin t)\, y'' + y = 1$, $y(1) = 1$, $y'(1) = 2$ certain to exist?

(a) $0 < t < 2\pi$ (b) $0 < t < \pi$ (c) $\pi < t < 2\pi$ (d) $-\infty < t < +\infty$
(e) cannot guarantee existence on any interval

$$y'' + \frac{1}{\sin t} y = \frac{1}{\sin t}$$

Avoid $\sin t = 0$



16. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}$. Find the matrix Q in the QR decomposition of A.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{2}{\sqrt{29}} \\ 0 & \frac{3}{\sqrt{29}} \\ 0 & \frac{4}{\sqrt{29}} \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 3/5 \\ 0 & 4/5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

(e) does not exist

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{9+16}}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

17. Find the least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (e) does not exist

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad A^T\vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

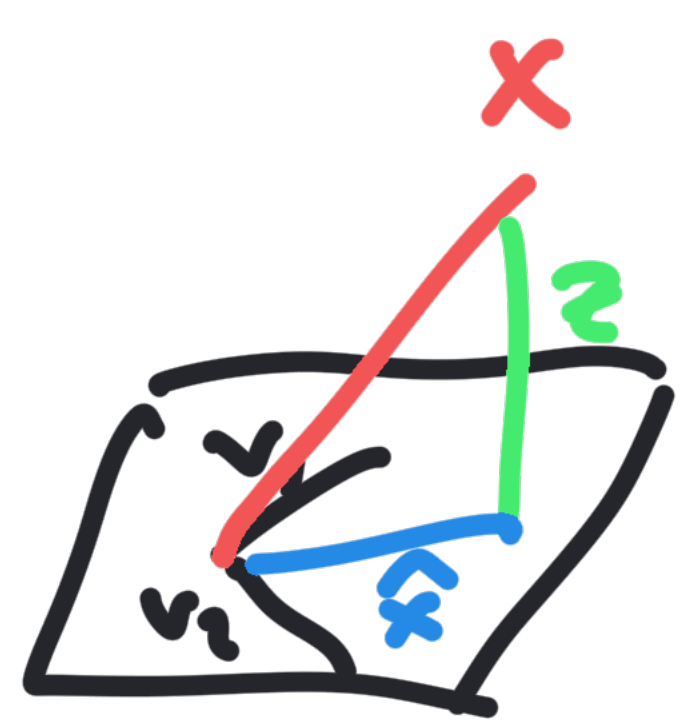
18. Find the distance between the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and the subspace in $\mathbb{R}^4$ spanned by the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{5}$ (e) 2

$$\vec{z} = \vec{x} - \frac{\vec{x}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{x}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} - \vec{0} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \|\vec{z}\| = \sqrt{4} = 2$$



19. The solution of the initial value problem $y' = \dfrac{1}{t\,e^y}$, $y(1) = 0$ is given implicitly by:

(a) $e^y = \ln t$ (b) $e^y = \ln t + 1$ (c) $\dfrac{t^2}{2} = -e^{-y} - \dfrac{1}{2}$ (d) $-e^{-y} = \ln t - 1$
(e) does not exist

$$\int e^y\,dy = \int \frac{dt}{t}$$

$$e^y = \ln t + C$$

$t=1$
$y=0$

$$1 = 0 + C$$

$$\boxed{C=1}$$

$$\boxed{e^y = \ln t + 1}$$

20. Find the general solution of the equation $t^2 y' + 4ty = 3$.

(a) $-\dfrac{3}{5}t^{-1} + Ct^4$ (b) $t^{-1} + Ce^{2t^2}$ (c) $t + Ct^4$ (d) $t^{-1} + Ct^{-4}$
(e) cannot be found explicitly using methods we learned

$$y' + \frac{4}{t}y = \frac{3}{t^2} \qquad \mu(t) = e^{\int 4/t\,dt}$$

$$= e^{4\ln t} = t^4$$

$$y(t) = \frac{\int t^4 \cdot 3/t^2\,dt}{t^4} = \frac{\int 3t^2\,dt}{t^4}$$

$$= \frac{t^3 + c}{t^4} = t^{-1} + ct^{-4}$$