Math 20580
Midterm 2
March 5, 2015

Name:________________________
Instructor:________________________
Section:________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b c d e

2. a b c d e

3. a b c d e

4. a b c d e

5. a b c d e

6. a b c d e

7. a b c d e

8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Part I: Multiple choice questions (7 points each)

1. Which of the following form a vector space?

   A. All continuous functions $f : [-1,1] \to \mathbb{R}$ with $\int_{-1}^{1} f(t)\,dt = 0$.

   B. All vectors of the plane in $\mathbb{R}^3$ defined by $x - y - z = -1$.

   C. All polynomials $p(t)$ with $p(0) = 0$.

   D. All continuous functions f on $\mathbb{R}$ with $f(1) = f(-1)$.

   (a) C,D only (b) A,C,D only (c) A,B,C and D (d) A,B only (e) A,B,D only

A, C, D are vector spaces.

B is NOT since 0 does not belong to

$\{(x,y,z) \mid x - y - z = -1\}$.

2. Let $\mathbb{P}_2$ be the space of all polynomials of degree less than or equal to two. What is the dimension of the subspace of $\mathbb{P}_2$ spanned by $\{1+t^2, 2-t+t^2, 1-t\}$?

   (a) 1 (b) 3 (c) 4 (d) 2 (e) 0

In term of st basis $(1, t, t^2)$. we need to compute rank of

$$\underset{\substack{\uparrow\ \ \ \ \ \ \ \ \uparrow\ \ \ \ \ \ \ \ \uparrow\\ 1+t^2\ \ 2-t+t^2\ \ 1-t}}{\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank $= 2$.

3. Which of the following statements are TRUE?

A. $\text{rank}(A) = \text{rank}(A^T)$.

B. An $n \times n$ matrix A is diagonalizable if and only if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

C. One can find n linearly independent vectors in $\mathbb{R}^n$ whose span is NOT all of $\mathbb{R}^n$.

D. Two matrices that are row equivalent always have the same eigenvalues.

(a) A,B,D only (b) B,C, D only (c) C,D only (d) B only (e) ✓ A, B only.

rank(A) = # of columns of A = # of rows of A^T = rank(A^T). So A. is TRUE.

B is clearly TRUE & C is clearly FALSE.

D is also FALSE. Since any invertible matrix is row eq. to I, but eigenvalues might be diff.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\0\\6\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\-2\\4\end{bmatrix}$$

Find $T\begin{bmatrix}1\\1\end{bmatrix}$.

(a) $\begin{bmatrix}2\\-2\\10\end{bmatrix}$ (b) $\begin{bmatrix}1\\-1\\5\end{bmatrix}$ (c) ✓ $\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ (d) $\begin{bmatrix}2\\2\\2\end{bmatrix}$ (e) cannot be determined from the given information

$$\left(\begin{array}{cc|c}1 & 3 & 1\\2 & 4 & 1\end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|c}1 & 3 & 1\\0 & -2 & -1\end{array}\right) \Rightarrow \begin{pmatrix}1\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\4\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}1\\1\end{pmatrix} = -\frac{1}{2}T\begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}T\begin{pmatrix}3\\4\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}2\\0\\6\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\-2\\4\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-1\\-1\end{pmatrix}.$$

5. If $\begin{bmatrix}1\\2\\1\end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix}4 & 0 & -1\\3 & 0 & 3\\2 & -2 & 5\end{bmatrix}$ then the corresponding eigenvalue is

(a) 3 (b) -2 (c) 1 (d) -1 (e) -3

$$\begin{pmatrix}4 & 0 & -1\\3 & 0 & 3\\2 & -2 & 5\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3\\6\\3\end{pmatrix} = 3\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$\Rightarrow$ eigenvalue $= 3$.

6. The eigenvalues of the matrix $A = \begin{bmatrix}1 & -1 & 2\\0 & 2 & 2\\0 & 1 & 3\end{bmatrix}$ are

(a) $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ (b) $\lambda = 4$ and $\lambda = 1$ (with multiplicity 2)
(c) $\lambda = 0$ and $\lambda = 1$ (with multiplicity 2) (d) $\lambda = 0$, $\lambda = 1$ and $\lambda = 2$
(e) The only real eigenvalue of A is 1.

$$\det(A - \lambda I) = \begin{vmatrix}1-\lambda & -1 & 2\\0 & 2-\lambda & 2\\0 & 1 & 3-\lambda\end{vmatrix}$$

go down this column.

$$= (1-\lambda)\left[(2-\lambda)(3-\lambda) - 2\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \lambda = 1, 1, 4.$$

7. Which of the following statements is **not** true for an invertible $n \times n$ matrix A?

(a) $\operatorname{rank} A = n$ (b) $\dim \operatorname{Row} A = n$ (c) $\dim \operatorname{Nul} A = 0$
(d) $A^T A^{-1}$ is invertible (e) $\lambda = 0$ is an eigenvalue of A

A is invertible $\Leftrightarrow$ if & only if $\dim \operatorname{Nul} = 0$

So only (e) is NOT true

8. Let $\mathcal{B} = \{1+t, 1-t^2, 1-t+t^2\}$ be a basis for the space of polynomials of degree at most 2. Find the coordinate vector $[p]_{\mathcal{B}}$ of $p(t) = 1 - 2t + t^2$.

(a) $\begin{bmatrix} 1/3 \\ -4/3 \\ 2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} -2/3 \\ 1/3 \\ 4/3 \end{bmatrix}$ (e) $\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

We need to solve

$$C_1(1+t) + C_2(1-t^2) + C_3(1-t+t^2) = 1-2t+t^2$$

or taking coordinate vectors with respect to $\{1, t, t$

$$C_1\begin{pmatrix}1\\1\\0\end{pmatrix} + C_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + C_3\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right) \xleftrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\underset{R_3 - R_2}{\xleftrightarrow{R_1 + R_2}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 3 & 4 \end{array}\right) \Rightarrow C_3 = 4/3$$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the following 3×4 matrix :

$$A = \begin{bmatrix} 3 & 6 & 0 & -2 \\ 1 & 2 & 1 & -2 \\ 0 & 0 & -3 & 4 \end{bmatrix}$$

(a) Find a basis for the row space of A.

$$\begin{pmatrix} 3 & 6 & 0 & -2 \\ 1 & 2 & 1 & -2 \\ 0 & 0 & -3 & 4 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 3 & 6 & 0 & -2 \\ 0 & 0 & -3 & 4 \end{pmatrix}$$

$$\xleftrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -3 & 4 \end{pmatrix} \xleftrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{Basis} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 4 \end{pmatrix} \right\}$$

(b) Based on your calculations above, what is the dimension of the null space of A^T. (**Hint:** The rows of A are the columns of A^T.)

A^T is 4×3,

$\text{rank}(A) = 2 = \text{rank}(A^T)$

$\Rightarrow \text{null}(A^T) = \#$ of columns of $A^T - 2$.

$= 3 - 2 = 1$

10. Consider two bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of coordinates matrix sending a $\mathcal{B}$-coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ to a $\mathcal{C}$-coordinate vector $[\mathbf{x}]_{\mathcal{C}}$.

Change of matrix.

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 2 & 4 & 5 & -1 \end{array}\right)$$

$$\xleftrightarrow{R_2-2R_1} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 0 & -2 & 3 & -3 \end{array}\right).$$

$$\xleftrightarrow{R_2/-2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 0 & 1 & -3/2 & 3/2 \end{array}\right)$$

$$\xleftrightarrow{R_1-3R_2} \left(\begin{array}{cc|cc} 1 & 0 & 11/2 & -7/2 \\ 0 & 1 & -3/2 & 3/2 \end{array}\right).$$

So $\underset{\mathcal{B}\to\mathcal{C}}{P} = \begin{pmatrix} 11/2 & -7/2 \\ -3/2 & 3/2 \end{pmatrix}$

11. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}$. Find an invertible matrix P such that

$A = PDP^{-1}$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$\Rightarrow \boxed{P = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}}$$

eigenvalues $1, -1, 3$.

$\underline{\lambda = 1}$

$$(A - \lambda I \mid 0) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{l} \varepsilon_1 + \varepsilon_2 - 3\varepsilon_3 = 0, \\ 2\varepsilon_2 = 2\varepsilon_3 \end{array} \qquad \vec{V}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Pick $\varepsilon_2 = 1$

$\underline{\lambda = -1}$

$$(A - \lambda I \mid 0) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ \cancel{1} & 3 & -3 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right); \quad \begin{array}{l} 2\varepsilon_1 = 0 \\ \varepsilon_2 = \varepsilon_3 \end{array}$$

$$\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 3}$

$$(A - \lambda I \mid 0) = \left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 1 & -1 & -3 & 0 \\ 1 & -1 & -3 & 0 \end{array}\right) \Rightarrow \begin{array}{l} \varepsilon_1 = 0 \\ -\varepsilon_2 = 3\varepsilon_3 \end{array} \qquad \vec{V}_3 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

12. Let $\mathbb{P}_2$ and $\mathbb{P}_3$ denote the spaces of polynomials of degree less than or equal to two and three respectively. Define a linear transformation $T : \mathbb{P}_3 \to \mathbb{P}_2$ by

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (a_0 + a_3) + (a_1 + a_3)x + (a_1 - a_2)x^2.$$

(a) Write down the standard bases $\mathcal{B}_2$ and $\mathcal{B}_3$ for $\mathbb{P}_2$ and $\mathbb{P}_3$ respectively.

$\mathcal{B}_2 = \{1, x, x^2\}$

$\mathcal{B}_3 = \{1, x, x^2, x^3\}.$

(b) Find the matrix for T relative to the bases $\mathcal{B}_2$ and $\mathcal{B}_3$.

$T(1) = 1 \Rightarrow \qquad [T(1)]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T(x) = x + x^2 \qquad [T(x)]_{\mathcal{B}_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$T(x^2) = -x^2. \qquad [T(x^2)]_{\mathcal{B}_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$T(x^3) = 1 + x. \qquad [T(x^3)]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

$\Rightarrow$ matrix : $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$