Name: **Version #1**

Instructor: ______________________

Math 20580, Final
December 12, 2016

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 2 hours.
- Be sure that your name and instructor's name are on the front page of your exam.
- Be sure that you have all pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
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Total ______________________

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1.(6pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ 3x + 8y \end{bmatrix}.$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the $\mathcal{B}$-matrix of T is a diagonal matrix.

(a) $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (e) $\left\{\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right\}$

2.(6pts) Imagine that you just got that great job and opened a retirement account (IRA) with initial balance 0 in which you plan to deposit money continuously at a rate of 15 thousand dollars per year for the next 40 years. If this account earns annual interest rate of 5% compounded continuously, find the amount in your account (**in thousands of dollars**) at the end of the 40 year period.

(a) $300(e^2 - 1)$ (b) $500(e^2 - 1)$ (c) $300(e^2 + 1)$

(d) $400(e^2 - 1)$ (e) $150(e^2 - 1)$

3.(6pts) The least squares solution $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ of the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

is given by

(a) $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ where $\hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

(b) None of the other answers are correct.

(c) $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ where $\hat{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \\ 12 \end{bmatrix}$.

(d) $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ where $\hat{\mathbf{b}} = \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$.

(e) $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ where $\hat{\mathbf{b}} = \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}$.

4.(6pts) If $y(t)$ is the solution to the initial value problem

$$y'' - 4y' + 5y = 0,\ y(0) = 0,\ y'(0) = 1$$

then find $y(\pi/2)$.

(a) $e^{\pi/2}$

(b) 0

(c) e^{π}

(d) $2e^{\pi}$

(e) π

5.(6pts) Find the maximum positive time T (lifespan) for which the solution to the initial value problem

$$\frac{dy}{dt} = \frac{1}{3}y^4, \quad y(0) = 0.1$$

is defined for all t with $0 \leq t < T$.

(a) $T = 1$ (b) $T = 3$ (c) $T = 10$

(d) $T = 1000$ (e) $T = 300$

6.(6pts) Let $\mathbb{P}_4$ be the space of all polynomials $a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$ with real coefficients. Consider the linear transformation $T : \mathbb{P}_4 \longrightarrow \mathbb{P}_4$ given by

$$T(p(t)) = 2p'(t).$$

Let A be the matrix for the linear transformation T with respect to the basis $\{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}_4$. Compute $\det A$.

(a) 12 (b) -1 (c) 1

(d) 0 (e) 24

7.(6pts) If the method of undetermined coefficients is to be used, then a suitable form for determining a particular solution y_p to the differential equation

$$y'' - 4y' + 4y = 2e^{2t} + \sin t + t$$

is given by:

(a) $y_p = At^2e^{2t} + (B\sin t + C\cos t) + (Dt + E)$

(b) $y_p = Ate^{2t} + C\sin t + Dt$

(c) $y_p = Ate^{2t} + (B\sin t + C\cos t) + (Dt + E)$

(d) $y_p = At^2e^{2t} + B\sin t + (Ct + D)$

(e) $y_p = Ae^{2t} + B\sin t + Ct$

8.(6pts) A mass m hanging at the end of a vertical spring causes an elongation L of the spring equal to 1/2 ft. Assume the mass is started in motion from the rest position with a velocity 16 ft/sec in the downward direction. What is the equation for the distance $u(t)$ (in feet) the mass is **below** the rest position at time t (in seconds)? (Use $g = 32\text{ft/s}^2$ for the acceleration due to gravity and neglect air resistance.)

(a) $u(t) = 2\sin 16t$ (b) $u(t) = 2\sin 4t$ (c) $u(t) = 2\sin 8t$

(d) $u(t) = \sin 4t$ (e) $u(t) = \sin 64t$

9.(6pts) Let

$$A = \begin{bmatrix} 0 & 1 & -1 & 3 & -2 \\ 0 & 3 & -3 & 2 & 4 \\ 0 & 2 & -2 & 3 & 1 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of the null space of A?

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

10.(6pts) Consider the system of linear equations

$$\begin{cases} (s-4)\,x + 2\,y = s \\ (2s-7)\,x + 4\,y = 1 \end{cases}$$

for x and y. What is the value of y in the solution?

(a) $-2s^2 + 8s - 4$

(b) $\dfrac{4s-2}{8s-30}$

(c) $2s-1$

(d) $\dfrac{s^2-4s+2}{8s-30}$

(e) $s^2 - 4s + 2$

11.(6pts) Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $r = \dim \operatorname{Col} A$, $s = \dim \operatorname{Nul} A$, and $t = \dim \operatorname{Row} A$, then

(a) $(r, s, t) = (2, 3, 2)$ (b) $(r, s, t) = (5, 0, 1)$ (c) $(r, s, t) = (3, 2, 4)$

(d) $(r, s, t) = (3, 2, 3)$ (e) $(r, s, t) = (2, 3, 3)$

12.(6pts) Which number is **not** an eigenvalue of the following matrix?

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 4 & 0 & 0 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

(a) 1 (b) -1 (c) 4

(d) 2 (e) -5

13.(6pts) Given that $y_1 = t$ and $y_2 = t^{-1}$ is a fundamental set of solutions for the differential equation $y'' + t^{-1}y' - t^{-2}y = 0$, the variation of parameters method gives a particular solution y_p to the corresponding nonhomogeneous equation

$$y'' + t^{-1}y' - t^{-2}y = t^{-1},\ x > 0,$$

of the form

$$y_p = tu_1 + t^{-1}u_2.$$

Which of the following functions can be $u_1(t)$?

(a) $u_1(t) = t^2 \ln t$ (b) $u_1(t) = t \ln t$ (c) $u_1(t) = -\frac{1}{2}t^2$

(d) $u_1(t) = \frac{1}{2} \ln t$ (e) $u_1(t) = 2t + 5t^{-1}$

14.(6pts) The Gram-Schmidt process applied to the basis $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \right\}$ yields an orthogonal basis that when normalized gives the **orthonormal** basis

(a) $\left\{ \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{110} \\ 6/\sqrt{110} \\ -12/\sqrt{110} \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 8/5\sqrt{5} \\ -16/5\sqrt{5} \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{105} \\ 4/\sqrt{105} \\ -8/\sqrt{105} \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{5} \\ 4/\sqrt{5} \\ -8/\sqrt{5} \end{bmatrix} \right\}$

15.(6pts) Which of the following numbers can **not** be the dimension of the null space of a 3×5 matrix?

(a) 2 (b) 5 (c) 3

(d) 4 (e) 1

16.(6pts) Given that $y_1(t) = e^t$ is a solution to the differential equation

$$ty'' - (2t + 1)\, y' + (t + 1)\, y = 0$$

then the reduction of order method gives a second solution of the form

$$y_2(t) = e^t u(t),$$

where $u(t)$ satisfies a simpler differential equations. Which of the following is this simpler differential equation?

(a) $u'' + u' = 0$ (b) $tu' - u = 0$ (c) $u' - 2t = 0$

(d) $tu'' - u' = 0$ (e) $u'' - 3u' + 2u = 0$

17.(6pts) Using the Existence and Uniqueness Theorem for second order linear differential equations, find the maximal interval of existence of the solution to the initial value problem

$$(t^3 - 9t)y'' - 8ty' + (t+4)y = t^2 - 9, \qquad y(2) = 5, \quad y'(2) = -1.$$

(a) $(0,3)$ (b) $(3,\infty)$ (c) $(-3,0)$

(d) $(-\infty,-3)$ (e) $(0,3)$

18.(6pts) For which value of k is the following linear system for x and y consistent:

$$\begin{cases} k\,x + y = 1-k \\ k\,x + 2\,y = 2-2k \\ (1+k)\,x + y = -k. \end{cases}$$

(a) $k=0$ (b) $k=1$ (c) $k=-1$

(d) $k=2$ (e) $k=-2$

19.(6pts) Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}.$$

Note that Col A = Col Q. Find R for the QR factorization of A.

(a) No such R exists.

(b) $R = \begin{bmatrix} 1 & 0 \\ -5/3 & 1/3 \end{bmatrix}$

(c) $R = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}$

(d) $R = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$

(e) $R = \begin{bmatrix} 1/3 & -5/3 \\ 0 & 1 \end{bmatrix}$

20.(6pts) The solution of the differential equation

$$(y - 3x^2 + 4) + (x + 4y^3 - 2y)\frac{dy}{dx} = 0$$

is given by the (implicit) relation:

(a) $yx - x^3 + 4x + y^4 = c$

(b) $yx - x^3 + 4x - y^2 = c$

(c) $y - x^2 + 4x + y^4 - y^2 = c$

(d) $yx - x^3 + 4x + y^4 - y^2 = c$

(e) $yx - x^3 + 4x = c$

21.(6pts) Let A be a 3×4 matrix and $\mathbf{b}$ be a column vector of length 3. Assume that the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. Let $B = [A|\mathbf{b}]$ be the augmented matrix, of size 3×5, of that linear system. Which of the following statements **must** be true?

(a) $\text{rank}(A) = 4$

(b) $\text{rank}(A) = 3$

(c) $\text{rank}(B) = \text{rank}(A) + 1$

(d) $\text{rank}(B) = \text{rank}(A)$

(e) $\mathbf{b} = \mathbf{0}$

22.(6pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\}$ be two basis of $\mathbb{R}^2$. Find the matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, i.e., the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(a) $\begin{bmatrix} 8 & -11 \\ 5 & -7 \end{bmatrix}$

(b) $\begin{bmatrix} 5/2 & 7/2 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 11 & -15 \\ -5 & 7 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & -5 \\ -1 & -3 \end{bmatrix}$

23.(6pts) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation given by counterclockwise rotation about the origin by $\pi/4$ (radians). Let A be the matrix of T with respect to the standard basis of $\mathbb{R}^2$. Which of the following matrices is equal to A^2?

(a) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

24.(6pts) Assume that the population $p(t)$ (in millions) of puffins (a type of seabird) at any time t (in years from now) is modeled by the differential equation

$$\frac{dp}{dt} = -0.15p(1 - \frac{p}{15})(1 - \frac{p}{100}).$$

Find the population of puffins in the distant future (i.e., $\lim_{t\to\infty} p(t)$) if currently there are 10 million puffins.

(a) 0 (b) 20 (c) 15

(d) 1 (e) 100

25.(6pts) Which of the following sets are vector spaces with the usual addition and scalar multiplication?

I. The set of all polynomials of the form $p(t) = a + t + bt^2$ for all $a, b \in \mathbb{R}$.

II. The set of vectors $\mathbf{v}$ in $\mathbb{R}^2$ such that $\mathbf{v} \cdot [3, 2]^T = 0$.

III. The set of all vectors $\mathbf{v}$ in $\mathbb{R}^3$ that are *not* scalar multiples of $[2, 1, 1]^T$.

IV. The set of functions that are solutions to $y'' + e^t y' - (\sec t)y = 0$.

(a) I, II and IV.

(b) III and IV.

(c) I and II.

(d) II and IV.

(e) I and III.