Name: Date: 4/23/2020

M20580 L.A. and D.E. Tutorial
Worksheet 10 solutions
Sections 2.4, 2.5, 2.6

1. Solve the initial value problem

$$y' = y^{1/3}, \quad y(0) = 1 \quad \text{for } t \geq 0.$$

Why is your solution unique?

Solution:

$y' = y^{1/3}$ is a separable, autonomous, first-order, non-linear differential equation.

$y^{-1/3}\,dy = dt \implies \frac{3}{2}y^{2/3} = t + c.$

The initial condition $y(0) = 1 \implies c = \frac{3}{2}$, so the solution is $y = \left(\frac{2}{3}t + 1\right)^{3/2}$.

This solution is unique by Theorem 2.4.2: $f(t, y) = y^{1/3}$ is continuous everywhere, and $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ is continuous whenever $y \neq 0$. So whenever (t_0, y_0) does not lie on the t-axis in the ty-plane (corresponding to the region $y \neq 0$), there is a unique solution to the differential equation $y' = y^{1/3}$ that passes through (t_0, y_0). In our case, $(0, 1)$ does not lie on the t-axis, so the solution above is unique.

2. Consider the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0 \quad \text{for } t \geq 0.$$

(a) What can you say about the existence or uniqueness of solutions to the initial value problem from Theorem 2.4.2?

Solution:

Since $(0, 0)$ lies on the t-axis where $\frac{\partial f}{\partial y}$ does not exist, we cannot apply Theorem 2.4.2 to guarantee uniqueness of solution. However, since f is continuous everywhere, we can still say that a solution exists (even though it may not be unique).

(b) Verify that the following differentiable functions, defined for an arbitrary positive t_0, are solutions to the initial value problem:

$$y = \begin{cases} 0, & \text{if } 0 \leq t < t_0, \\ \pm[\frac{2}{3}(t - t_0)]^{3/2}, & \text{if } t \geq t_0. \end{cases}$$

In particular, there are infinitely many solutions.

Solution:

It is clear that $y(0) = 0$ is satisfied.

For $0 \leq t \leq t_0$, $y' = 0 = y^{1/3}$.

For $t \geq t_0$, $y' = \pm[\frac{2}{3}(t - t_0)]^{1/2} = y^{1/3}$.

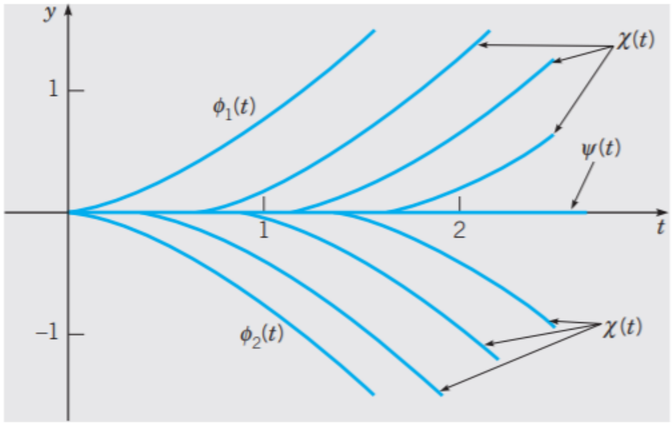

FIGURE 2.4.1 Several solutions of the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

Figure 2.4.1 (taken from the textbook) shows a few of these solutions given in part (b).

Note that If (t_0, y_0) is any point not on the t-axis, then Theorem 2.4.2 guarantees that there is a unique solution of the differential equation $y' = y^{1/3}$ passing through (t_0, y_0).

(c) Why is there no solution to the initial value problem that passes through $(1, 1)$?

Solution:

Since $(1, 1)$ does not lie on the t-axis, there is a unique solution to $y' = y^{1/3}$ that passes through $(1, 1)$.

By guessing that the solution would look like the ones given in part (b) and doing some reverse engineering to solve for t_0, we find that the solution is

$$y = \begin{cases} 0, & \text{if } 0 \le t < -1/2, \\ [\frac{2}{3}(t + \frac{1}{2})]^{3/2}, & \text{if } t \ge -1/2. \end{cases}$$

In particular, $y(0) \neq 0$, so there is no solution to the initial value problem that passes through $(1, 1)$.

(d) Find the solution to the initial value problem that passes through $(2, 1)$.

Solution:

Since $(2, 1)$ does not lie on the t-axis, there is a unique solution to $y' = y^{1/3}$ that passes through $(2, 1)$.

Using the condition $y(2) = 1$ to solve for t_0 in part (b), we find that $t_0 = 1/2$, so the function

$$y = \begin{cases} 0, & \text{if } 0 \le t < 1/2, \\ [\frac{2}{3}(t - \frac{1}{2})]^{3/2}, & \text{if } t \ge 1/2. \end{cases}$$

is the unique solution to the initial value problem that passes through $(2, 1)$.

(e) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

Solution:

By the same reasoning as in parts (c) and (d), a solution $y(t)$ to the initial value problem has to be of the form given in part (b).

This means that $y(2)$ can take any value $\pm[\frac{2}{3}(2-t_0)]^{3/2}$ for any choice of $t_0 \geq 0$. Therefore $y(2)$ is in the range $|y(2)| \leq (4/3)^{3/2} \approx 1.54$.

3. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$dy/dt = \alpha y(1-y), \quad y(0) = y_0,$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

(a) Find the equilibrium points for the differential equation and determine whether each is asymptotically stable, semistable, or unstable.

Solution:

$y' = \alpha y(1-y)$ is a separable, autonomous, first-order, non-linear differential equation.

The equilibrium points are at $y = 0, 1$.

Since $f(y) = \alpha y(1-y)$ is positive when $0 < y < 1$, $y = 0$ is unstable while $y = 1$ is asymptotically stable.

(b) Solve the initial value problem and verify that the conclusions you reach in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

Solution:

Rearrange the differential equation:

$$\frac{1}{y(1-y)}\, dy = \alpha\, dt.$$

Rewrite the fraction on the left-hand side in partial fractions:

$$\frac{1}{y} + \frac{1}{1-y}\, dy = \alpha\, dt.$$

Integrate both sides:

$$\ln|y| - \ln|1-y| = \alpha t + C \implies \ln\left|\frac{y}{1-y}\right| = \alpha t + C \implies \frac{y}{1-y} = Ce^{\alpha t}.$$

Note that we could remove the absolute value signs since $\frac{y}{1-y}$ is positive in the range that we care about.

The initial condition $y(0) = y_0$ gives $C = \frac{y_0}{1-y_0}$.

Rearrange to get the solution $y = \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}}$.

Thus we see that $y \to 1$ as $t \to \infty$, independent of the value of y_0 (as long as it is in the range $(0, 1]$).

4. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \tag{1}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = -\alpha xy. \tag{2}$$

(a) Determine y at any time t by solving Eq. (1) subject to the initial condition $y(0) = y_0$.

Solution:

The solution is $y = y_0 e^{-\beta t}$. Note that $y \to 0$ as $t \to \infty$, so the proportion of carriers vanishes over time.

(b) Use the result of part (a) to find x at any time t by solving Eq. (2) subject to the initial condition $x(0) = x_0$.

Solution:

Part (a) turns Eq. (2) into $\frac{dx}{dt} = -\alpha x y_0 e^{-\beta t}$.

$$\int \frac{1}{x}\, dx = -\alpha y_0 \int e^{-\beta t}\, dt \implies \ln x = \frac{\alpha y_0}{\beta} e^{-\beta t} + C.$$

The initial condition $x(0) = x_0$ gives $C = \ln x_0 - \frac{\alpha y_0}{\beta}$, so the solution is

$$x = x_0 e^{\alpha y_0 (e^{-\beta t} - 1)/\beta}.$$

(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

Solution:

As $t \to \infty$, the proportion x of susceptibles goes to $x_0 e^{-\alpha y_0/\beta}$.

Summing up, over the long term, the proportion y of carriers vanishes while the proportion x of susceptibles goes to $x_0 e^{-\alpha y_0/\beta}$, so this is the proportion population that escapes the epidemic.

5. Consider the differential equation

$$3y^2 - 4x(y^3 + 1) + xy(2 - 3xy)y' = 0.$$

Is it exact? If not, does it have an integrating factor? Even better, does it have an integrating factor that is a function of x or y alone?

Solution:

The differential equation is not exact: $M = 3y^2 - 4x(y^3 + 1) \implies M_y = 6y - 12xy^2$, while $N = xy(2 - 3xy) \implies N_x = 2y - 6xy^2$. Since $M_y \neq N_x$, the differential equation is not exact.

Now we find an integrating factor.

$M_y - N_x = 6y - 12xy^2 - 2y + 6xy^2 = 2y(2 - 3xy) \implies \dfrac{M_y - N_x}{N} = \dfrac{2y(2 - 3xy)}{xy(2 - 3xy)} = \dfrac{2}{x}$
is a function of x alone.

So we have an integrating factor $\mu(x)$ which is a function of x alone, and which satisfies $\dfrac{d\mu}{dx} = \dfrac{M_y - N_x}{N}\mu = \dfrac{2}{x}\mu$.
We can solve this separable first-order linear differential equation and choose for instance the integrating factor $\mu = x^2$. ($\mu = -x^2$ also works.)

6. Consider the differential equation

$$(y - 3x^2 + 4) + (x + 4y^3 - 2y)\frac{dy}{dx} = 0.$$

(a) Is the above differential equation exact?

Solution:

$M = y - 3x^2 + 4 \implies M_y = 1.$

$N = x + 4y^3 - 2y \implies N_x = 1.$

$M_y = N_x \implies$ exact.

(b) Find the solution of the above differential equation.

Answer: $yx - x^3 + 4x + y^4 - y^2 = c$

Solution:

We want to find $\psi(x, y)$ such that $\psi_x = y - 3x^2 + 4$ and $\psi_y = x + 4y^3 - 2y$.

The first condition $\psi_x = y - 3x^2 + 4$ implies $\psi(x, y) = \int y - 3x^2 + 4 \, dx = xy - x^3 + 4x + h(y)$. This also gives $\psi_y = x + h'(y)$, which we will now use to find $h(y)$.

The second condition $\psi_y = x + 4y^3 - 2y$ implies $\psi_y = x + h'(y) = x + 4y^3 - 2y \implies h'(y) = 4y^3 - 2y \implies h(y) = y^4 - y^2$.

Therefore $\psi(x, y) = xy - x^3 + 4x + y^4 - y^2$, and the solution to the differential equation is $xy - x^3 + 4x + y^4 - y^2 = c$ for any constant c.