Coordinates and Change of Basis(Solutions)

2/27/2020

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases for $\mathbb{R}^3$ where the vectors are the following:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathbf{b}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

and

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{c}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \mathbf{c}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Let *std* be the standard basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$. Note that for all the vectors given above, those coordinates are with respect to the standard basis (i.e. $\mathbf{b}_1 = [\mathbf{b}_1]_{\text{std}}$)

1. What is the change of basis matrix from $\mathcal{B}$ to standard, $P_{\mathcal{B}}$?

   This is the matrix which inputs e_i and outputs b_i. This matrix is just the matrix whose columns are b_i, i.e. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$.

2. What is the change of basis matrix from $\mathcal{C}$ to standard, $P_{\mathcal{C}}$?

   Just as above, this is the matrix whose i^{th} column is c_i, i.e. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$.

3. What are the coordinates of $\mathbf{b}_1$ with respect to the $\mathcal{B}$ basis? (i.e. what is $[\mathbf{b}_1]_{\mathcal{B}}$?)

   They are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, by definition.

4. What is $P_{\mathcal{B}} [\mathbf{b}_1]_{\mathcal{B}}$?(note this is matrix multiplication)

   This is

$$\mathcal{B} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b_1$$

5. Do you recognize your answer to Question 3? Which vector in the preamble is that vector? If you are unsure of what is going on here, repeat Question 3 replacing $\mathbf{b}_2$ and $\mathbf{b}_3$ for $\mathbf{b}_1$.

   This is the vector b_1!

6. Repeat Questions 3 and 4 replacing all small b's and big B's with the appropriate case of the letter C.

   You can do it. You'll get that for instance that $P_{\mathcal{C}}[c_2]_{\mathcal{C}} = c_2$.

7. Compute the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Do this two ways

   - Compute it as $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$ This the matrix

   $$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/6 & 1/2 & 0 \\ 1/3 & 0 & 1 \end{pmatrix}$$

   - Compute it by row reducing the augmented matrix $[P_{\mathcal{C}}|P_{\mathcal{B}}]$ to get the desired matrix on the right hand side after row reducing to reduced row echelon form.

     You can do it. You'll get the same answer as above because the row operations are effected by matrix multiplication. When you do the same row operation on $P_{\mathcal{C}}$ and $P_{\mathcal{B}}$, you are multiplying the matrix on the left and the right by the same matrix. Hence when you row reduce to make $P_{\mathcal{C}}$, i.e. multiplying by $P_{\mathcal{C}}^{-1}$, you end up multiplying $P_{\mathcal{B}}$ by $P_{\mathcal{C}}^{-1}$.

   - Did you get the same answer both ways? If not, do it again.

     Yes dammit.

8. Using the calculations for the coordinates of vectors from Question 3 compute $P_{C\leftarrow B}\,[\mathbf{b}_1]_{\mathcal{B}}$.

   This is

   $$\begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/6 & 1/2 & 0 \\ 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/6 \\ 1/3 \end{pmatrix}$$

9. The answer to Question 8 should be the coordinates of $\mathbf{b}_1$ with respect to $\mathcal{C}$. Check this taking the weighted sum of $[c_1]_{std}, [c_2]_{std}, [c_3]_{std}$ using the coordinates as weights. Does your answer agree with question 8? If not, then do this again (maybe do earlier problems again also).

   We calculate that

   $$(1/2)[c_1]_{std} + (1/6)[c_2]_{std} + (1/3)[c_3]_{std} = \begin{pmatrix} 1/2 + 1/6 + 1/3 \\ 2/6 - 1/3 \\ 3/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

10. What magic is happening in problem 9 that makes this work? If you are unsure, repeat problem 9 for $\mathbf{b}_2$ and $\mathbf{b}_3$. Also repeat problem 9 for these if you were not able to get it to work on the first round.

    The coordinates of b_1 with respect to the basis $\mathcal{C}$ are defined so that the weighted sum of the c_i's with respect to the coordinates give you back b_1. So this better happen when we explicitly compute and it does.

11. Now for the fun, pick your favorite vector in $\mathbb{R}^3$, keep in mind that the vector you write down will be given with coordinates with respect to the standard basis for $\mathbb{R}^3$. My favorite is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, you cannot use mine but I wanted to give an example. Keep this new vector in mind, call it x. Now do the following:

    (a) Compute the coordinates of x with respect to $\mathcal{B}, [x]_\mathcal{B}$, by solving the system of equations given by the augmented matrix $[P_\mathcal{B}|\mathbf{x}]$ (why does this work?).

    This will work because the result will be $P_\mathcal{B}^{-1}x$. Since P_B is the matrix which goes converts B coordinates to standard coordinates, P_B^{-1} is the matrix which goes from standard coordinates to $\mathcal{B}$-coordinates.

    When I compute the solution of the system, where I choose my favorite vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, I get $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

    (b) Compute the same thing as in 11 (a) by computing $P_\mathcal{B}^{-1}\mathbf{x}$.

    We have $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

    (c) Do your answers to (a) and (b) match? If not do them both again until they do. Why are these computations equivallent?

    These computations are equivalent because the effect of row reducing is is multiplying everything by the same matrix(the matrix that effects the row operation). When we row reduce to get the identity matrix to the left of the divider, we are baisically multiplying whatever is on the right of the divider by the inverse of the matrix on the left divider. Hence in both cases you actually are computing the same thing.

    (d) Double check your answers in (a) and (b) by doing the following. Do computation with respect to the standard bases. Take the coordinates in (a) and (b), use them as in a weighted sum of the vectors $[\mathbf{b}_1]_{std}, [\mathbf{b}_2]_{std}, [\mathbf{b}_3]_{std}$ and check if you get the original vector x.

    We have that $2[b_1]_{std} - 1[b_2]_{std} + 0[b_3]_{std} = \begin{pmatrix} 2-1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(e) Now compute the coordinates of x with respect to $\mathcal{C}$, $[x]_{\mathcal{C}}$, first do this in the ways given in (a) and (b), and do the checking steps in step (c) and (d)

When I did this, the coordinates of x with respect to $\mathcal{C}$ were $\begin{pmatrix} 1/2 \\ -1/6 \\ 2/3 \end{pmatrix}$.

(f) Now the fun begins, take the change of basis from $\mathcal{B}$ to $\mathcal{C}$ that you computed in problem 7, and compute $P_{C \leftarrow B}[\mathrm{x}]_{\mathcal{B}}$ Is this what you got in part (e), does it match?

We check that

$$\begin{pmatrix} 1/2 \\ -1/6 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/6 & 1/2 & 0 \\ 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

They are the same. Good.

If you feel uncertain, repeat exercise 11 for as many vectors as you like. Also try going from $\mathcal{C}$ to $\mathcal{B}$. Also try playing around with other bases and doing all the problems from scratch. Have fun!