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M20580 L.A. and D.E. Tutorial
Worksheet 8
Sections 6.4, 6.5/1.1, 1.2

1. Apply the Gram-Schmidt process to find an orthogonal basis for the column space of the following matrix

$$\begin{bmatrix} 3 & -7 & 8 \\ -1 & 5 & -2 \\ 1 & 1 & 1 \\ 3 & -5 & 1 \end{bmatrix}$$

You need not normalize your basis.

Let $\vec{x}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -7 \\ 5 \\ 1 \\ -5 \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} 8 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

$\mathrm{Col}(A) = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}.$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 8 \\ -2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

Thus, an orthogonal basis for Col (A) is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}.$$

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2. Find a **least squares solution** to the system

$$\begin{bmatrix} 1 & -1 & -5 \\ 6 & 1 & 0 \\ 1 & -5 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \end{bmatrix} \leftarrow \vec{b}$$

Note that the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of the coefficient matrix A form an **orthogonal** basis for Col A.

Method 1: Since $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ forms an orthogonal basis for Col(A), we can compute $\hat{b} = \text{proj}_{\text{Col}(A)} \vec{b}$ by

$$\hat{b} = \frac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \frac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 + \frac{\vec{b}\cdot\vec{a}_3}{\vec{a}_3\cdot\vec{a}_3}\vec{a}_3$$

$$= \frac{2}{3}\vec{a}_1 + 0\vec{a}_2 + \frac{1}{3}\vec{a}_3$$

$\hat{x}$:= the least squares solution.

$$= [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} = A \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

Method 2: ~~Let~~ Solve

$A^T A \hat{x} = A^T b$, and get $\hat{x} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$.

$\Downarrow$ $\Uparrow$

Solve

$$\begin{bmatrix} 1 & 6 & 1 & 4 \\ -1 & 1 & -5 & 0 \\ -5 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -5 \\ 6 & 1 & 0 \\ 1 & -5 & 1 \\ 4 & 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 6 & 1 & 4 \\ -1 & 1 & -5 & 0 \\ -5 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 54 & & \\ & 27 & \\ & & 27 \end{bmatrix} \hat{x} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

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3. Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}.$$

Note that Col A = Col Q. Find R for the QR factorization of A.

Note columns of Q are orthonormal,

$$R = Q^T A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}.$$

4. Solve the following initial value problem

$$\frac{dA}{dt} = 0.05A + 15, \quad A(0) = 0.$$

$$\frac{dA}{dt} = 0.05A + 15 \iff \frac{dA}{dt} = 0.05(A + 300)$$

$$\iff \frac{dA}{dt} \cdot \frac{1}{A+300} = 0.05$$

$$\iff \frac{d}{dt}\left[\ln(A+300)\right] = 0.05$$

$$\iff \ln(A+300) = 0.05t + C.$$

$$\iff A + 300 = e^{0.05t + C} = e^{C} e^{0.05t}$$

$$\iff A = k e^{0.05t} - 300$$

By $A(0) = 0 \iff k - 300 = 0 \iff k = 300$

Thus $A(t) = 300 e^{0.05t} - 300.$