Math 20580 Name:____________________________

Midterm 1 Instructor:____________________________

February 16, 2023 Section:____________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c ~~d~~ e
2. a b c d ~~e~~
3. a ~~b~~ c d e
4. a b c d ~~e~~
5. a b ~~c~~ d e
6. ~~a~~ b c d e
7. a ~~b~~ c d e
8. a b ~~c~~ d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. For the matrix
$$A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$$
determine the sum $A^{-1} + A^T$ between the inverse of A and the transpose of A.

(a) $\begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & -8 \\ 8 & 21 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{2\cdot 2 - 1\cdot 5}\begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$
$$A^{-1} + A^T = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$$

2. Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ denote (in this order) the columns of the matrix
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 5 \end{bmatrix}.$$
Which of the following sets of vectors are linearly independent?

(I) $\{\vec{v}_1, \vec{v}_2\}$ (II) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (III) $\{\vec{v}_2\}$ (IV) $\{\vec{v}_2, \vec{v}_3\}$

(a) III only (b) I and III only (c) I and II only (d) II and IV only (e) III and IV only

$\vec{v}_2 = 2\vec{v}_1$ So I, II dependent

$\vec{v}_2 \neq \vec{0}$ so $\underline{\text{III}}$ independent

$\vec{v}_2, \vec{v}_3$ not scalar multiples, so $\underline{\text{IV}}$ independent

3. Consider linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrices

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad [S] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

What is the matrix of the composition $T \circ S$?

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4. Let $M_{2,3}$ denote the vector space of 2×3 matrices. Which among the following subsets of $M_{2,3}$ is a subspace?

I. The set of all 2×3 matrices whose columns sum to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

II. The set of all 2×3 matrices whose entries are all non-negative.

III. $\left\{ \begin{bmatrix} t & t+s & s \\ 0 & s+2t & 0 \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$

IV. $\left\{ \begin{bmatrix} t & 1 & 0 \\ 0 & s & 1 \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$

(a) III and IV only (b) IV only (c) I, III, and IV only

(d) II and IV only (e) III only

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is NOT in I, IV

II is NOT closed under scalar multiplication by -1

III: $\left\{ t \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} + s \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

is a subspace!

5. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 + 2x_4 \\ x_1 + 2x_2 + x_3 \\ x_2 - x_3 + x_4 \end{bmatrix}$$

Determine the standard matrix of T.

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & -3 \\ 4 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & 0 \\ 0 & x_2 & -x_3 & x_4 \end{bmatrix}$

(e) none of the above

$$\| $$

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}$$

6. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix}$$

For which value of t does the vector $\vec{v}_3$ belong to $\text{Span}(\vec{v}_1, \vec{v}_2)$?

(a) $t = 2$ only (b) all $t \geq 0$ (c) $t = 1$ and $t = -1$ (d) $t = 0$ only

(e) $t = 1$ only

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & t \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & t-1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{array}\right]$$

don't want pivot in 3rd column

$$\Rightarrow \boxed{t - 2 = 0}$$

7. Which of the following sets is a basis of $\mathbb{R}^2$?

(I) $\left\{\begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}8\\4\end{bmatrix}\right\}$ (II) $\left\{\begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$ (III) $\left\{\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\right\}$ (IV) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$

(a) I, II, III only (b) II, III, IV only (c) I, II only

(d) III and IV only (e) I, III, IV only

$\left\{\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix}\right\}$ basis of $\mathbb{R}^2$

$\Leftrightarrow ad-bc \neq 0$

So I NOT basis

$\underline{\text{II}}, \underline{\text{III}}, \underline{\text{IV}}$ basis

8. Let A be a 2×6 matrix. Describe all the possible values for the nullity of A (the dimension of the null space of A)?

(a) 0,1,2,3 (b) 2,3,4 (c) 4,5,6 (d) 2, 4, 6 (e) 1,2,3,4,5.

rank A = 0, 1 or 2 (each row contains at most one pivot)

rank A + nullity A = 6

Therefore, nullity A = 6, 5 or 4

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the linear system

$$\begin{cases} x_1 + x_3 = 1 \\ 2x_1 + 2x_3 + x_4 = 1 \\ x_1 + x_2 + 2x_3 = 2 \\ 2x_2 + x_3 + x_4 = 1 \end{cases}$$

(a) Write down the augmented matrix of the system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

(b) Determine the solution set of the linear system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_4 \to R_4 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

move R_2 to the bottom

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

REF

$-x_3 + x_4 = -1 \Rightarrow \boxed{x_3 = 0}$

$\boxed{x_4 = -1}$

$x_2 + x_3 = 1 \Rightarrow \boxed{x_2 = 1}$

$x_1 + x_3 = 1 \Rightarrow \boxed{x_1 = 1}$

Unique solution

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

10. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 - R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\underbrace{\qquad}_{I_3}$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}$$

11. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \quad \underline{R_1 \leftrightarrow R_2}$$

(a) Find a basis $\mathcal{B}$ for Col(A) (the column space of A).

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} \quad \text{REF} \qquad \text{pivots in columns } 1,2,3$$

$$\underset{\substack{\uparrow \\ \text{columns } 1,2,3 \text{ in } A}}{\mathcal{B} =} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(b) Find a basis $\mathcal{R}$ for Row(A) (the row space of A).

non-zero rows in REF (all of them)

$$\mathcal{R} = \{ [1\ 4\ 2\ 1], [0\ 2\ 0\ 0], [0\ 0\ 3\ 1] \}$$

(c) For the basis $\mathcal{B}$ found in (a), determine the coordinate vector $[\vec{v}]_{\mathcal{B}}$ if $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Observe that $\vec{v}$ = last column of A

$\mathcal{B}$ = first three columns of A

Think: A is augmented matrix

$$\cdots \left[\begin{array}{ccc:c} 1 & 4 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right] \begin{array}{l} \to x_1 = 1/3 \\ \to x_2 = 0 \\ \to x_3 = 1/3 \end{array} \Rightarrow \boxed{[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}}$$

12. Consider the bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}, \qquad \mathcal{C} = \left\{ \overset{\vec{c}_1}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \overset{\vec{c}_2}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \overset{\vec{c}_3}{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$ (recall that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \cdot [\vec{x}]_{\mathcal{C}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\boxed{\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}}$$

(b) If $\vec{v}$ is a vector in $\mathbb{R}^3$ with $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, determine $[\vec{v}]_{\mathcal{B}}$ and $\vec{v}$.

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{v} = 0 \cdot \vec{c}_1 + 1 \cdot \vec{c}_2 + 0 \cdot \vec{c}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(also: $\vec{v} = 0 \cdot \vec{b}_1 + 1 \cdot \vec{b}_2 + 2 \cdot \vec{b}_3$)