

1. Which of the following statements is always true?
- I. The null space of an $m \times n$ matrix is a subspace of \mathbb{R}^m .
 - II. If the set $B = \{v_1, \dots, v_n\}$ spans a vector space V and $\dim V = n$, then B is a basis for V .
 - III. The rank of a $m \times n$ matrix A is the dimension of the column space of A .

- (a) II and III (b) I (c) I and III (d) I and II and III (e) None are true.

Solution. II and III are true.

- I. The null space is a subspace of \mathbb{R}^n so this is false.
 - II. This is the basis theorem.
 - III. This is the definition of rank.
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2. Let \mathbb{P}_n be the space of polynomial functions of degree at most n . Which of the following is NOT true?

- (a) The set of all polynomials of degree less than 2 with $f(0) = 1$ is a subspace of \mathbb{P}_3 .
- (b) \mathbb{P}_5 is a subspace of \mathbb{P}_8 .
- (c) If $T : \mathbb{P}_4 \rightarrow \mathbb{R}$ is a linear transformation, then the kernel of T is a subspace of \mathbb{P}_4 .
- (d) If \mathcal{B} is a basis for \mathbb{P}_2 , then \mathcal{B} is linearly independent in \mathbb{P}_3 .
- (e) $\mathcal{B} = \{t - 1, t + 1, t^2\}$ is a basis for \mathbb{P}_2 .

Solution. The set of all polynomials of degree less than 2 with $f(0) = 1$ is not closed under addition, so it cannot be a subspace.

3. Which of the following is NOT a linear transformation?

- (a) $T : \mathbb{R} \rightarrow \mathbb{R}$ where $T(x) = x^2$
- (b) $D : \mathbb{P}_5 \rightarrow \mathbb{P}_4$ defined by $D(f(t)) = 2f'(t)$, where f' denotes the derivative of f .
- (c) $T : C \rightarrow C$ where $T(f(t)) = (t + 1)f(t)$ and C is the space of continuous real valued functions.
- (d) $T : V \rightarrow V$ that sends each vector to itself (i.e. $T(\mathbf{x}) = \mathbf{x}$ for all \mathbf{x}).
- (e) $T : \mathbb{P}_3 \rightarrow \mathbb{R}$ where $T(f(t)) = 5f(1)$

Solution. If $T(x) = x^2$, then $T(5x) = 25x^2 \neq 5x^2 = 5T(x)$, so T is not a linear transformation

4. The transformation $T : \mathbb{P}_3 \rightarrow \mathbb{R}$ defined by $T(f(t)) = f(2)$ is a linear transformation. What is a basis for the kernel of T ?

- (a) $\{t^3 - 8, t^2 - 4, t - 2\}$ (b) $\{1, 2, 4, 8\}$ (c) $\{1, t, t^2, t^3\}$ (d) $\{[1]\}$ (e) $\{t, t^2, t^3\}$

Solution. Using the basis $\{1, t, t^2, t^3\}$ for \mathbb{P}_3 , we see that the matrix for the transformation is $\begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}$. The matrix is in row reduced echelon form, so variables 2,3,4 are free.

Therefore, a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ or $\{t^3 - 8, t^2 - 4, t - 2\}$.

5. $\mathcal{B} = \{1+t, 1-t, 2+t+3t^2\}$ is a basis for \mathbb{P}_2 . Find the coordinate vector of $p(t) = t^2$ with respect to \mathcal{B} .

- (a) $\begin{bmatrix} -1/2 \\ -1/6 \\ 1/3 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 \\ 1/6 \\ -1/3 \end{bmatrix}$ (c) $[1]$ (d) $\begin{bmatrix} 1+t \\ 1-t \\ 2+t+3t^2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solution. Row reduce $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$ thus the coordinate vector

is $\begin{bmatrix} -1/2 \\ -1/6 \\ 1/3 \end{bmatrix}$.

6. Find a basis for the eigenspace of $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$ with eigenvalue 3.

- (a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
 (e) 3 is not an eigenvalue of the matrix

Solution. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$, then the solutions of $(A - 3I)\mathbf{x} = \mathbf{0}$ are all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

7. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & * & -1 \end{bmatrix},$$

where $*$ can be any number. What are the eigenvalues of A ?

- (a) 2, 3, -1 (b) 2, 1, -3 (c) It is not possible to tell without knowing $*$ (d) 2, 0, 0
 (e) $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

Solution. The matrix is lower triangular, so the characteristic polynomial is $(2-t)(3-t)(-1-t)$, and the eigenvalues are 2, 3, -1.

8. What is the characteristic polynomial of the matrix $\begin{bmatrix} -3 & 4 \\ 1 & 2 \end{bmatrix}$?

- (a) $\lambda^2 + \lambda - 10$ (b) $\lambda^2 + \lambda - 6$ (c) $\lambda^2 - \lambda + 10$ (d) $(-3 - \lambda)(2 - \lambda)$ (e) $\lambda^2 - \lambda + 6$

Solution. The characteristic polynomial is

$$\det \begin{bmatrix} -3 - \lambda & 4 \\ 1 & 2 - \lambda \end{bmatrix} = (-3 - \lambda)(2 - \lambda) - 4 = \lambda^2 + \lambda - 10.$$

9. The eigenvalues of a 3×3 matrix A are $-1, 1$ with 1 having multiplicity 2 . Which of the following statements is true?

- (a) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for all \mathbf{b} .
- (b) The matrix A has a 3-dimensional null space.
- (c) The diagonal entries of A are $-1, 1, 1$.
- (d) The matrix A can be diagonalized.
- (e) The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution. If none of the eigenvalues are zero then the matrix is invertible.

10. Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ be two bases of \mathbf{R}^2 .

(a) Find the change of coordinates matrix ${}_{\mathcal{C} \leftarrow \mathcal{B}}P$.

(b) Suppose that \mathbf{v} is a vector in \mathbf{R}^2 such that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Compute $[\mathbf{v}]_{\mathcal{C}}$.

Solution.

(a) We row reduce:

$$\begin{bmatrix} -1 & -2 & 2 & -3 \\ 2 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & -1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & -5 & 2 \end{bmatrix},$$

so the change of basis matrix is

$${}_{\mathcal{C} \leftarrow \mathcal{B}}P = \begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix}.$$

(b)

$$[\mathbf{v}]_{\mathcal{C}} = {}_{\mathcal{C} \leftarrow \mathcal{B}}P [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

11. Consider the transformation $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ given by $T(f(t)) = (3t - 2)f'(t)$, where $f'(t)$ is the derivative of $f(t)$. For example,

$$T(t^2 - 3t + 1) = (3t - 2)(2t - 3) = 6t^2 - 13t + 6.$$

The function T is a linear transformation. You do not need to show this.

- (a) Calculate the matrix A of T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$ for \mathbb{P}_2 .
- (b) Find a basis for the kernel (null space) of T consisting of elements of \mathbb{P}_2 . What is its dimension?
- (c) Find a basis for the range of T consisting of elements of \mathbb{P}_2 . What is its dimension?

Solution.

(a) We compute

$$T(1) = 0,$$

$$T(t) = 3t - 2,$$

$$T(t^2) = (3t - 2)(2t) = 6t^2 - 4t.$$

It follows that the matrix of T with respect to the basis \mathcal{B} is

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 6 \end{bmatrix}.$$

(b) Row reducing A we have

$$\begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the null space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, which corresponds to $\{1\}$ in \mathbb{P}_2 . The dimension is 1.

(c) The range of T is the same as the column space of A , which has basis $\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} \right\}$, which corresponds to $\{-2 + 3t, -4t + 6t^2\}$ in \mathbb{P}_2 . The dimension is 2.

12. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$. Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$.

You do not need to compute P^{-1} .

Solution. First compute the characteristic polynomial:

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 6 & 2-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)(2-\lambda) - (2-\lambda)6 = (2-\lambda)(\lambda^2 - 5\lambda) = \lambda(2-\lambda)(\lambda-5).$$

The eigenvalues are 0, 2, 5.

We next find corresponding eigenvectors.

eigenvalue 0:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis of the null space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$.

eigenvalue 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the null space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

eigenvalue 5:

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

We can choose D, P so that

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 0 & 2 \end{bmatrix}.$$
