Math 20580 Name:____________________

Midterm 3 Instructor:____________________

November 16, 2021 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. $\boxed{\cancel{a}}$ $\boxed{b}$ $\boxed{c}$ $\boxed{d}$ $\boxed{e}$
2. $\boxed{a}$ $\boxed{\cancel{b}}$ $\boxed{c}$ $\boxed{d}$ $\boxed{e}$
3. $\boxed{a}$ $\boxed{b}$ $\boxed{\cancel{c}}$ $\boxed{d}$ $\boxed{e}$
4. $\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{d}$ $\boxed{\cancel{e}}$
5. $\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{d}$ $\boxed{\cancel{e}}$
6. $\boxed{a}$ $\boxed{b}$ $\boxed{\cancel{c}}$ $\boxed{d}$ $\boxed{e}$
7. $\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{\cancel{d}}$ $\boxed{e}$
8. $\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{\cancel{d}}$ $\boxed{e}$

Multiple Choice. ______________

9.

10.

11.

12.

Total. ______________

Part I: Multiple choice questions (7 points each)

1. Consider the line L spanned by the vector $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The distance from the vector $\vec{x} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$ to the line L is

(a) $\sqrt{5}$ (b) $\sqrt{45}$ (c) $2\sqrt{3}$ (d) 5 (e) $\sqrt{50}$

$$\mathrm{perp}_L(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= \begin{bmatrix} -7 \\ 1 \end{bmatrix} - \frac{15}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} \qquad \|\mathrm{perp}_L(\vec{x})\| = \sqrt{5}$$

2. Consider the matrices

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 & 11 \\ 3 & 6 & -3 & 1 & 15 \\ -1 & -2 & 1 & 2 & 2 \\ 4 & 8 & -4 & 4 & 28 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where B is the reduced row echelon form of A. A basis for the orthogonal complement of the row space of A is given by

$\mathrm{row}(A)^{\perp} = \mathrm{null}(A)$

(a) $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -3 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ -4 \\ 4 \\ 28 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 4 \\ 6 \\ -2 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 11 \\ 15 \\ 2 \\ 28 \end{bmatrix} \right\}$

$\dim(\mathrm{null}(A)) = 3$

$\mathrm{null}(A) \subseteq \mathbb{R}^5$

3. Consider the line L spanned by the unit vector $\vec{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$, and let $\text{proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation that sends a vector to its orthogonal projection onto the line L. The standard matrix of the transformation proj_L is

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} 3 & 5 \\ 5 & -4 \end{bmatrix}$ (c) $\frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$ (d) $\frac{1}{5}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$

(e) none of the above

$$A = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \qquad A^TA = [1]$$

$$AA^T = \frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$$

$$[\text{proj}_L]_{std} = A(A^TA)^{-1}A^T$$

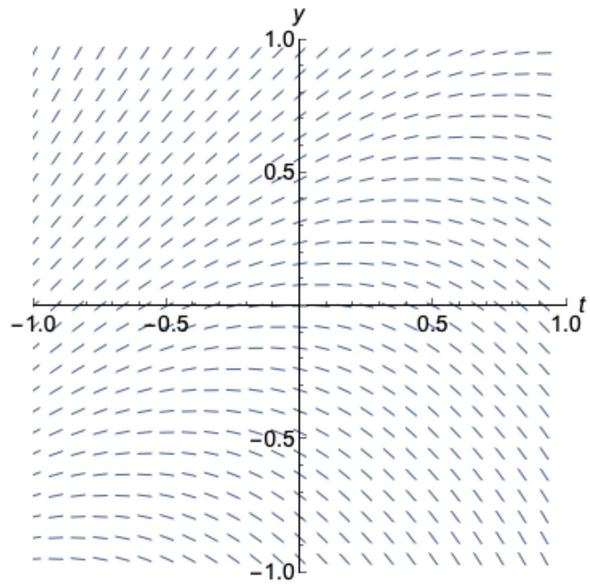
4. Which of the following functions is the solution of the equation $y' = ty$ with $y(0) = 1$?

(a) t (b) $e^{\cos t}$ (c) $t^2/2$ (d) e^t (e) $e^{t^2/2}$

$y(0) \neq 1$ $\quad y(0) \neq 1$ $\quad y(0) \neq 1$

$y' = y$ $\quad y' = ty$

5. Determine $f(t, y)$ if the differential equation $\dfrac{dy}{dt} = f(t, y)$ has direction field (the value of t is measured on the horizontal axis, and the value of y on the vertical axis)



- Depends on both t + y.
- Is 0 when y + t are zero.
- Positive + Negative values

~~(a)~~ $\sin(y)$ ~~(b)~~ $t^2 + y^2$ ~~(c)~~ y ~~(d)~~ $-t$ (e) $y - t$

6. Consider the autonomous equation $y' = y(y-1)(y-2)(y-3)$ with initial condition $y(0) = 2.99$. Without solving the equation explicitly, find the limit $\lim_{t\to+\infty} y(t)$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) ∞

+
3
−
2
+
1
−
0
+

$y'(y=2.99) < 0$

$\Rightarrow$ function decreases to next lowest crit. point.

7. The differential equation

$$\frac{dy}{dt} + ty^2 = 0$$

is

(a) ~~an equation of order 2~~ (b) ~~a partial differential equation~~ (c) ~~linear~~
(d) separable (e) none of the above

$$\frac{dy}{dt} = f(t)\,g(y) \qquad f(t) = t \qquad g(y) = y^2$$

8. Which of the following sets of vectors are orthogonal?

(I) $\left\{ \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} \right\}$ (II) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}$

(III) $\left\{ \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ (IV) $\left\{ \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right\}$

(a) ~~I only~~ (b) ~~I and III only~~ (c) ~~II only~~ (d) II and IV only (e) none of these

$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = 2 \neq 0 \Rightarrow (I) \text{ not orthog.}$$

$$\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -30 \neq 0 \Rightarrow (III) \text{ not orthog.}$$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ -3 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for W.

1st obtain orthogonal basis:

$$\vec{v}_1 = \vec{w}_1 \qquad \vec{v}_2 = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{w}_2 - \frac{0}{12} \vec{w}_1 = \vec{w}_2$$

$$\vec{v}_3 = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 3 \\ 3 \\ 3 \\ -3 \end{bmatrix} - \frac{12}{12} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{12}{12} \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

2nd, Normalize

$$\boxed{\vec{q}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad \vec{q}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} \quad \vec{q}_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}}$$

(b) Find the QR decomposition of the matrix A with columns $\vec{w}_1, \vec{w}_2, \vec{w}_3$.

$$Q = \boxed{\frac{1}{\sqrt{12}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix}} \quad \underbrace{\frac{1}{\sqrt{12}} \begin{bmatrix} 12 & 0 & 12 \\ 0 & 12 & 12 \\ 0 & 0 & 12 \end{bmatrix}}_{\text{"}} = \boxed{\sqrt{12} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$R = Q^T A = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ 3 & -1 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

$$A^TA = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{9} \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix}$$

$$\vec{x} = (A^TA)^{-1} A^T\vec{b} = \frac{1}{9} \begin{bmatrix} 9 \\ -9 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

$$A\vec{x} = \text{proj}_{Col(A)}(\vec{b})$$

$$\boxed{= \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}$$

11. Consider the differential equation

$$y + (2x - 4y^2) \cdot \frac{dy}{dx} = 0.$$

(a) Explain why the equation is not exact.

$M(x,y) = y$ $\quad M_y = 1 \quad N_x = 2$

$N(x,y) = 2x - 4y^2$ $\quad \boxed{M_y \neq N_x}$

(b) Find an integrating factor μ which only depends on the variable y.

We look for $\mu(y)$ s.t. $\quad \mu'(y)\, y = \mu$

$$(\mu M)_y = (\mu N)_x = \mu N_x \Rightarrow \int \frac{\mu'(y)}{\mu} dy = \int \frac{1}{y} dy$$

$$= \mu'(y) M + \mu M_y$$

$$= \mu'(y) y + \mu = 2\mu$$

$$\Rightarrow \ln(\mu) = \ln(y)$$

$$\boxed{\mu = y}$$

(c) Write down the implicit solution which satisfies the initial condition $y(1) = 1$.

Look for f s.t.

$$\frac{\partial f}{\partial x} = y^2 = \mu M \qquad \frac{\partial f}{\partial y} = (2x - 4y^2)\, y = \mu N$$

$$f = y^2 x + g(y)$$

$$2yx + g'(y) = 2yx - 4y^3$$

$$\Rightarrow g(y) = -y^4 + C$$

$$y^2 x - y^4 = C,$$

$$1^2 \cdot 1 - 1^4 = C = 0 \Rightarrow \boxed{y^2 x - y^4 = 0}$$

12. (a) Find the solution of the initial value problem

$$\begin{cases} t^2y' + 4ty = 3 \\ y(1) = -1 \end{cases}$$

Linear $\rightsquigarrow$ integrating factors

$$y' + \frac{4}{t}y = \frac{3}{t^2} \qquad \mu(t) = e^{\int \frac{4}{t}\,dt} = e^{4\ln t} = t^4$$

$$y(t) = \frac{1}{t^4}\int \frac{3}{t^2}t^4\,dt = \frac{1}{t^4}\left(t^3 + c\right)$$

$$= \frac{1}{t} + \frac{c}{t^4} \qquad \boxed{\frac{1}{t} - \frac{2}{t^4}}$$

$$y(1) = 1 + c = -1 \Rightarrow c = -2$$

(b) Find the maximal interval on which the solution to the initial value problem above is defined.

$\boxed{(0, \infty)}$, since we need t to be non-zero and to include $t = 1$.