Math 20580 Name:________________________
Final Exam Instructor:________________________
December 10, 2021 Section:________________________

Calculators are NOT allowed. You will be allowed 180 minutes to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1.	a	b	c	d	e	11.	a	b	c	d	e
2.	a	b	c	d	e	12.	a	b	c	d	e
3.	a	b	c	d	e	13.	a	b	c	d	e
4.	a	b	c	d	e	14.	a	b	c	d	e
5.	a	b	c	d	e	15.	a	b	c	d	e
6.	a	b	c	d	e	16.	a	b	c	d	e
7.	a	b	c	d	e	17.	a	b	c	d	e
8.	a	b	c	d	e	18.	a	b	c	d	e
9.	a	b	c	d	e	19.	a	b	c	d	e
10.	a	b	c	d	e	20.	a	b	c	d	e

1. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$

(e) none of the above

2. Let M be the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

Which of the following are eigenvalues of M?

I. 0 II. 1 III. 2 IV. 3

(a) I, II, and IV only (b) I, II, and III only (c) II, III, and IV only

(d) all of them (e) none of them

3. Let L be a line through the origin in $\mathbb{R}^{2021}$. What is the dimension of $L^{\perp}$?

(a) 2021 (b) 2020 (c) 1997 (d) 1 (e) none of these

4. Consider the exact first-order equation

$$\left(\frac{y}{x} + 6x\right) + (\ln(x) - 2)\, y' = 0.$$

Which of the following is the general implicit solution to this equation?

(a) $y \ln(x) + 3x^2 = C$ (b) $\dfrac{y^2}{2x} + 6xy = C$ (c) $(\ln(x) - 1)x - 2x = C$

(d) $y \ln(x) - 2y = C$ (e) $y \ln(x) + 3x^2 - 2y = C$

5. Let A be a 2×2 matrix with $\det(A) = 7$. Which of the following is true?

(a) A is NOT invertible
(b) A is invertible and $\det(A^{-1}) = 7$
(c) $\det(A^T) = 1/7$
(d) $A^T A$ is NOT invertible
(e) A is invertible and $\det(A^{-1}) = 1/7$

6. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2, and consider its basis $\mathbb{B} = \{t^2 + 2t - 1, 2t + 1, 1\}$. With respect to $\mathbb{B}$, the coordinates of $t^2 + 6t + 4$ are:

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (e) none of the above

7. Consider the initial-value problem

$$\sin(t)y'' + 3y = \tan(t), \quad y(1) = 1.$$

Which of the following is the largest interval on which a solution is guaranteed to exist?

(a) $(0, \pi/2)$ (b) $(0, \pi)$ (c) $(\pi/2, \pi)$ (d) $(0, \infty)$ (e) $(-\infty, \infty)$

8. Let S be a subspace of $\mathbb{R}^3$ of dimension 2. Which of the following sets of vectors could be a basis for S?

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ (e) none of these

9. What is the dimension of the row space of $A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ -3 & -5 & -1 & -12 \\ 2 & 3 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

10. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then b_{32} is:

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

11. Find the solution of the initial value problem

$$\begin{cases} y'' + y' - 2y = 0, \\ y(0) = 3, \ y'(0) = -6. \end{cases}$$

(a) $2e^{-3t}$ (b) $e^t + 2e^{-2t}$ (c) $3e^{-2t}$ (d) $-6e^t + 3e^{-2t}$ (e) $2e^t + e^{-2t}$

12. Consider the equation

$$y'' - 2y' + 2y = 0.$$

Let y_1 be the solution satisfying $y_1(0) = 1$, $y_1'(0) = 2$, and let y_2 be the solution satisfying $y_2(0) = 3$, $y_2'(0) = 4$. Using Abel's formula, find the Wronskian $W(y_1, y_2)$.

Hint: you can find the constant in Abel's formula by computing $W(y_1, y_2)$ at $t = 0$ using the initial conditions on y_1, y_2.

(a) 0 (b) $-2e^{2t}$ (c) $-2e^{-2t}$ (d) $4e^{2t}$ (e) $-2e^{-t^3/3}$

13. Consider the differential equation $y'' - 2y' + y = 2xe^x$. By the method of undetermined coefficients, a particular solution will have the form

(a) $(Ax^3 + Bx^2)e^x$ (b) $(Ax + B)e^x$ (c) Axe^x
(d) Axe^{-x} (e) $A\sin(x) + B\cos(x)$

14. Find the solution of the initial value problem

$$\begin{cases} y + 3xy' = 0, \quad x > 0 \\ y(1) = 1 \end{cases}$$

(a) $3x - 2$ (b) $x^{-1/3}$ (c) x^2 (d) $x^{-2/3}$ (e) there is no solution

15. Which of the following can *not* be the rank of a 7×5 matrix?

(a) 0 (b) 1 (c) 2 (d) 5 (e) 7

16. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 0 & 7 \end{bmatrix}$. Find the matrix Q in the QR decomposition of A.

(a) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{57}} \\ \frac{-1}{\sqrt{2}} & \frac{-2}{\sqrt{57}} \\ 0 & \frac{7}{\sqrt{57}} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 7 \end{bmatrix}$

(e) does not exist

17. Which of the following describes the least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ only (b) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ only (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ only

(d) infinitely many solutions (e) no solutions

18. Which formula describes the general solution of the differential equation

$$t^2y'' - 4ty' + 6y = 0, \ t > 0$$

given the fact that $y_1(t) = t^2$ is a solution of this equation?

(a) $c_1t^2 + c_2t^3$ (b) $c_1t^2 + c_2$ (c) $c_1t^2 + c_2te^t$ (d) $c_1t\ln(t) + c_2t^2$ (e) $c_1t + c_2t^2$

19. Consider the differential equation $y'' + y = \cos^2(x)$. The functions

$$y_1 = \cos(x) \quad \text{and} \quad y_2 = \sin(x)$$

form a fundamental set of solutions for the associated homogeneous equation. Variation of parameters produces a solution to the nonhomogeneous ODE of the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Up to a constant of integration, what is u_1?

(a) $-\dfrac{1}{3}\sin^3(x)$ (b) $\cos(x)$ (c) $-\dfrac{1}{2} - \dfrac{1}{4}\sin(2x)$ (d) $\dfrac{1}{3}\cos^3(x)$
(e) none of the above

20. Find the general solution of the equation $y' + t^2 y = t^2$.

(a) $C + e^{-t^3/3}$ (b) $1 + Ce^{t^3/3}$ (c) $t^2 + Ce^{-t}$ (d) $1 + Ce^{-t^3/3}$
(e) cannot be found explicitly using methods we learned