Math 20580 Name:________________________________
Final Exam Instructor:____________________________
December 10, 2021 Section:______________________________

Calculators are NOT allowed. You will be allowed 120 minutes to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1.	a	b	c	~~d~~	e	11.	a	b	~~c~~	d	e
2.	~~a~~	b	c	d	e	12.	a	~~b~~	c	d	e
3.	a	~~b~~	c	d	e	13.	~~a~~	b	c	d	e
4.	a	b	c	d	~~e~~	14.	a	~~b~~	c	d	e
5.	a	b	c	d	~~e~~	15.	a	b	c	d	~~e~~
6.	~~a~~	b	c	d	e	16.	a	b	~~c~~	d	e
7.	~~a~~	b	c	d	e	17.	a	b	c	~~d~~	e
8.	~~a~~	b	c	d	e	18.	~~a~~	b	c	d	e
9.	a	b	c	~~d~~	e	19.	a	b	c	~~d~~	e
10.	a	b	c	~~d~~	e	20.	a	b	c	~~d~~	e

1. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$
(e) none of the above

$$\left[\begin{array}{cc:cc} 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc:cc} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc:cc} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$\underset{C \leftarrow B}{P}$

2. Let M be the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

Which of the following are eigenvalues of M?

I. 0 ✓ II. 1 ✓ ~~III. 2~~ IV. 3 ✓

(a) I, II, and IV only (b) I, II, and III only (c) II, III, and IV only

(d) all of them (e) none of them

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)(2-\lambda) - 2 \right]$$

$$= (1-\lambda)(3\lambda - \lambda^2)$$

$$= (1-\lambda)(3-\lambda)\cdot\lambda \implies \lambda = 0, 1, 3 \text{ are eigenvalues}$$

3. Let L be a line through the origin in $\mathbb{R}^{2021}$. What is the dimension of $L^{\perp}$?

(a) 2021 (b) 2020 (c) 1997 (d) 1 (e) none of these

$$\dim(L^{\perp}) = 2021 - \dim L$$
$$= 2021 - 1$$
$$= 2020$$

4. Consider the exact first-order equation

$$\left(\frac{y}{x} + 6x\right) + (\ln(x) - 2)\, y' = 0.$$

Which of the following is the general implicit solution to this equation?

(a) $y\ln(x) + 3x^2 = C$ (b) $\dfrac{y^2}{2x} + 6xy = C$ (c) $(\ln(x) - 1)x - 2x = C$

(d) $y\ln(x) - 2y = C$ (e) $y\ln(x) + 3x^2 - 2y = C$

$$\begin{cases} M = \frac{y}{x} + 6x = \frac{\partial f}{\partial x} \\ N = \ln x - 2 = \frac{\partial f}{\partial y} \end{cases} \Rightarrow f(x,y) = \int M\,dx = y\ln x + 3x^2 + g(y)$$

$$\frac{\partial f}{\partial y} = \ln x + g'(y) = \ln x - 2$$

$$\Rightarrow g(y) = -2y \quad \Rightarrow f(x,y) = y\ln x + 3x^2 - 2y$$

Implicit solution: $y\ln x + 3x^2 - 2y = C$

5. Let A be a 2×2 matrix with $\det(A) = 7$. Which of the following is true?

(a) A is NOT invertible
(b) A is invertible and $\det(A^{-1}) = 7$
(c) $\det(A^T) = 1/7$
(d) $A^T A$ is NOT invertible
(e) A is invertible and $\det(A^{-1}) = 1/7$

6. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2, and consider its basis $\mathcal{B} = \{t^2 + 2t - 1, 2t + 1, 1\}$. With respect to $\mathcal{B}$, the coordinates of $t^2 + 6t + 4$ are:

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (e) none of the above

Relative to basis $\{t^2, t, 1\}$ for $\mathbb{P}_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot [t^2+6t+4]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 6 \\ -1 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [t^2+6t+4]_{\mathcal{B}}$$

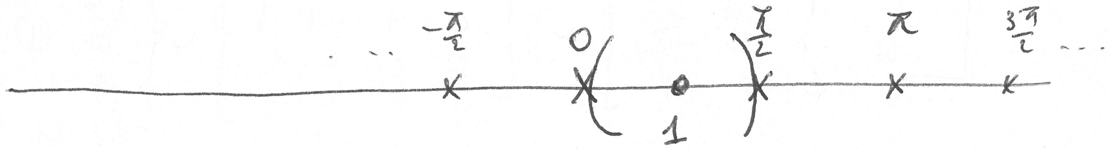
7. Consider the initial-value problem

$\frac{\sin t}{\cos t}$ ← need $\cos t \neq 0$ so $t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$$\sin(t)y'' + 3y = \tan(t), \quad y(1) = 1.$$

Which of the following is the largest interval on which a solution is guaranteed to exist?

(a) $(0, \pi/2)$ (b) $(0, \pi)$ (c) $(\pi/2, \pi)$ (d) $(0, \infty)$ (e) $(-\infty, \infty)$



Contains because $y(1) = \ldots$

Need $\sin t \neq 0$
(coefficient of y'')

so $t \neq 0, \pm\pi, \pm 2\pi, \ldots$

8. Let S be a subspace of $\mathbb{R}^3$ of dimension 2. Which of the following sets of vectors could be a basis for S?

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \right\}$ ~~(b)~~ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \right\}$

~~(d)~~ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ (e) none of these

2x

Need 2 linearly independent vectors!

9. What is the dimension of the row space of $A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ -3 & -5 & -1 & -12 \\ 2 & 3 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then b_{32} is:

(a) −2 (b) −1 (c) 0 (d) 1 (e) 2

$$b_{32} = \frac{C_{23}}{\det A} = \frac{1}{1} = 1$$

$$C_{2,3} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(-1) = 1$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = (-1)(-2+1) = 1$$

11. Find the solution of the initial value problem

$$\begin{cases} y'' + y' - 2y = 0, \\ y(0) = 3, \ y'(0) = -6. \end{cases} \rightarrow m^2+m-2=0 \quad (m+2)(m-1)=0$$

$m_1 = -2, \ m_2 = 1$

(a) $2e^{-3t}$ (b) $e^t + 2e^{-2t}$ (c) $3e^{-2t}$ (d) $-6e^t + 3e^{-2t}$ (e) $2e^t + e^{-2t}$

General solution $y = C_1 e^{-2t} + C_2 e^t$

$\overset{t=0}{\Rightarrow} \boxed{3 = C_1 + C_2}$

$y' = -2C_1 e^{-2t} + C_2 e^t \overset{t=0}{\Rightarrow} \boxed{-6 = -2C_1 + C_2}$ $\Rightarrow$ $C_1 = 3$, $C_2 = 0$

$\Downarrow$

$y = 3e^{-2t}$

12. Consider the equation

$$y'' - 2y' + 2y = 0.$$

Let y_1 be the solution satisfying $y_1(0) = 1$, $y_1'(0) = 2$, and let y_2 be the solution satisfying $y_2(0) = 3$, $y_2'(0) = 4$. Using Abel's formula, find the Wronskian $W(y_1, y_2)$.

Hint: you can find the constant in Abel's formula by computing $W(y_1, y_2)$ at $t = 0$ using the initial conditions on y_1, y_2.

(a) 0 (b) $-2e^{2t}$ (c) $-2e^{-2t}$ (d) $4e^{2t}$ (e) $-2e^{-t^3/3}$

$$W(y_1, y_2)(t) = C \cdot e^{\int 2dt} = C \cdot e^{2t}$$

at $t=0$

$$C = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$$

So $W(y_1, y_2)(t) = -2e^{2t}$

13. Consider the differential equation $y'' - 2y' + y = 2xe^x$. By the method of undetermined coefficients, a particular solution will have the form

(a) $(Ax^3 + Bx^2)e^x$ (b) $(Ax + B)e^x$ (c) Axe^x
(d) Axe^{-x} (e) $A\sin(x) + B\cos(x)$

$Y = x^s \cdot (Ax+B)\, e^x$ for $s \in \{0,1,2\}$

So No terms are solutions of homogeneous eqn!

Homogeneous eqn:

$y'' - 2y' + y = 0$

has $FSS = \{e^x, xe^x\}$

$\Rightarrow \underline{\underline{s=2}}$

$Y = x^2(Ax+B)\, e^x = (Ax^3 + Bx^2)\, e^x$

14. Find the solution of the initial value problem

$$\begin{cases} y + 3xy' = 0, \quad x > 0 \\ y(1) = 1 \end{cases}$$

(a) $3x - 2$ (b) $x^{-1/3}$ (c) x^2 (d) $x^{-2/3}$ (e) there is no solution

$3x \frac{dy}{dx} = -y$ Separable

$\int \frac{dy}{y} = \int \frac{-dx}{3x}$

$\ln|y| = -\frac{1}{3} \ln x + C$

$\Rightarrow y = x^{-1/3} \cdot K$

$y(1) = 1 \Rightarrow K = 1$, so $\boxed{y(x) = x^{-1/3}}$

15. Which of the following can *not* be the rank of a 7×5 matrix?

(a) 0 (b) 1 (c) 2 (d) 5 (e) 7

rank ≤ 5

16. Let $A = \begin{array}{c} \begin{array}{cc} \vec{w_1} & \vec{w_2} \\ \| & \| \end{array} \\ \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 0 & 7 \end{bmatrix} \end{array}$. Find the matrix Q in the QR decomposition of A.

(a) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{57}} \\ \frac{-1}{\sqrt{2}} & \frac{-2}{\sqrt{57}} \\ 0 & \frac{7}{\sqrt{57}} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 7 \end{bmatrix}$

(e) does not exist

Gram-Schmidt: $\vec{v_1} = \vec{w_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\|\vec{v_1}\| = \sqrt{2}$

$\vec{v_2} = \vec{w_2} - \frac{\vec{w_1} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1}$

$= \begin{bmatrix} 2 \\ -2 \\ 7 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$, $\|\vec{v_2}\| = 7$

Orthonormal basis $\left\{ \frac{1}{\|\vec{v_1}\|} \vec{v_1}, \frac{1}{\|\vec{v_2}\|} \vec{v_2} \right\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

So $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$

17. Which of the following describes the least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ only (b) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ only (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ only

(d) infinitely many solutions (e) no solutions

$$A^T A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad A^T \vec{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 2 \\ -2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -2 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

free so infinitely many solutions

18. Which formula describes the general solution of the differential equation

$$t^2 y'' - 4ty' + 6y = 0, \ t > 0 \qquad \rightsquigarrow \quad y'' - \frac{4}{t} y' + \frac{6}{t^2} y = 0 \qquad P(t) = -\frac{4}{t}$$

given the fact that $y_1(t) = t^2$ is a solution of this equation?

(a) $c_1 t^2 + c_2 t^3$ (b) $c_1 t^2 + c_2$ (c) $c_1 t^2 + c_2 t e^t$ (d) $c_1 t \ln(t) + c_2 t^2$ (e) $c_1 t + c_2 t^2$

$y_2 = v \cdot y_1$, $\quad V = \int \frac{e^{-\int P(t)\,dt}}{y_1^2(t)}\,dt$

v not constant

$$= \int \frac{e^{\int \frac{4}{t} dt}}{t^4}\,dt = \int \frac{e^{4 \ln t}}{t^4}\,dt$$

$$= \int 1\,dt = t$$

So

$y_2 = t \cdot t^2$

$= t^3$

Get $y = c_1 t^2 + c_2 t^3$

$g(x)$

19. Consider the differential equation $y'' + y = \cos^2(x)$. The functions

$$y_1 = \cos(x) \quad \text{and} \quad y_2 = \sin(x)$$

form a fundamental set of solutions for the associated homogeneous equation. Variation of parameters produces a solution to the nonhomogeneous ODE of the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Up to a constant of integration, what is u_1?

(a) $-\frac{1}{3}\sin^3(x)$ (b) $\cos(x)$ (c) $-\frac{1}{2} - \frac{1}{4}\sin(2x)$ (d) $\frac{1}{3}\cos^3(x)$
(e) none of the above

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$$

$$u_1' = \frac{-y_2 g}{W} = -\sin x \cos^2 x \Rightarrow u_1 = \int -\sin x \cos^2 x \, dx$$

$$\underset{du = -\sin x\, dx}{\overset{u = \cos x}{=}} \int u^2 du = \frac{u^3}{3} = \frac{\cos^3 x}{3}$$

20. Find the general solution of the equation $y' + t^2 y = t^2$ $g(t)$
(a) $C + e^{-t^3/3}$ (b) $1 + Ce^{t^3/3}$ (c) $t^2 + Ce^{-t}$ (d) $1 + Ce^{-t^3/3}$
(e) cannot be found explicitly using methods we learned

Integrating factor $\mu(t) = e^{\int t^2 dt} = e^{t^3/3}$

Solution $y(t) = \dfrac{\int \mu(t) g(t) dt}{\mu(t)} = \dfrac{\int e^{t^3/3} \cdot t^2 \, dt}{e^{t^3/3}}$

$$= \frac{e^{t^3/3} + C}{e^{t^3/3}} = 1 + Ce^{-t^3/3}$$