Name: SOLUTIONS Date: 03/02/2023

M20580 L.A. and D.E. Tutorial
Quiz 5

1. Let $V = \{A \in \mathcal{M}_{2\times 2} \mid A^T = -A\}$. A matrix $A \in V$ is called skew symmetric. Find a basis $\mathcal{B}$ for V, the vector space of all 2×2 skew symmetric matrices.

   Hint: Consider a 2×2 matrix A and analyze the effect of the condition $A^T = -A$ on the entries of A.

   Solution: Let $A \in \mathcal{M}_{2\times 2}$, so $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Notice that

   $$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \text{and} \quad -A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$$

   If $A^T = -A$, then $-a = a$, $c = -b$, $b = -c$ and $d = -d$. These restrictions simplify to $a = 0$, $d = 0$ and $c = -b$, so the matrix A actually looks like

   $$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   From the last equality, we see that V is spanned by $\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Since $\mathcal{B}$ has one nonzero element, it is linearly independent and is a basis for V.

2. Let $V = \mathbb{R}^3$ and consider the set $W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$. Determine whether W is a subspace of V.

   Solution: Recall that every subspace W contains the zero vector $\mathbf{0}$. If we take

   $$\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

   this forces $a = 0$ and $b = 0$, but $a + b + 1 = 1 \neq 0$. Therefore, $\mathbf{0}$ is not an element of W and W is not a subspace of V.