Name: SOLUTIONS Date: 04/06/2023

M20580 L.A. and D.E. Tutorial
Quiz 8

1. Consider the following vectors:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

They form a basis for $\mathbb{R}^2$. Apply the Gram-Schmidt process to obtain an orthogonal basis (You must use Gram-Schmidt for full credit).

Solution: We will denote the vectors resulting from the Gram-Schmidt process $\mathbf{v}_1$ and $\mathbf{v}_2$. To begin with, we just let $\mathbf{v}_1 = \mathbf{u}_1$. To determine the second vector, we do:

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

Therefore, the final answer is:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

2. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Solution: We first solve for $A^T A$ and its inverse:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (A^T A)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we have:

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$