Name: Date: 01/19/2023

M20580 L.A. and D.E. Tutorial
Worksheet 1

1. Use either Gaussian or Gauss-Jordan eliminiation algorithm to solve the following linear systems. What is the rank of each?

a) $\begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 0 \\ 4x_1 - x_2 + x_3 = 4 \end{cases}$ b) $\begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \end{cases}$

Solution:

a) Let's row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow{R_2-2R_1,\ R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}\right] \xrightarrow{-1/5\, R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{array}\right] \xrightarrow{R_3+9R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{array}\right] \rightarrow \begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ x_2 - 7/5x_3 = 18/5 \\ 2/5x_3 = 2/5 \end{cases}$$

We see from the REF that the rank is three. Now use back substitution to solve the system:

$$x_3 = 1, \rightarrow\ x_2 = \frac{18+7}{5} = 5, \rightarrow\ x_1 = 9 - 10 + 3 = 2.$$

Thus the solution of the first system is $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

b) As in the previous bullet, row reduce the matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_2+R_1,\ R_3-2R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right] \xrightarrow{R_3+10R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -x_2 - x_3 = 0 \end{cases}$$

Now we see from the REF that the rank is two. Let $t := x_3$ be a parameter. Using back substitution, we find that $x_2 = -t,\quad x_1 = -3t + 2t = -t$. Thus the solution is $\mathbf{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

2. Given the augmented matrices of some linear systems, determine how many solutions they have, if any; also, if the corresponding linear system is consistent, determine its rank and the number of free variables.

a) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$ b) $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

c) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$

Solution:

a) The matrix is not yet in a REF, but after $R_4 - 2R_2$ it is in a REF:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{R_4 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the system is consistent. The rank of the system is 3 and the number of variables is 4, hence one of the variables must be a free variable. This implies the system has infinitely many solutions.

b) As in the previous case, we need to row reduce the matrix first (you can also spot an issue in this example without any computation):

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The third row represents the equation $0 = -3$, which is a contradiction. Therefore, the system is inconsistent.

c) We row reduce the matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_4 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

The system is consistent. The rank is 4 and the number of variables is 4, hence there are no free variables, and thus the solution is unique.

3. Determine if a given vector $\mathbf{v}$ is a linear combination of vectors $\mathbf{u}_i$. If yes, write $\mathbf{v}$ as a linear combination of those vectors.

a) $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$. b) $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Hint: a vector $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ if you can find scalars α_1 and α_2 such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$. The question about existence of a linear combination can be translated into the question about consistency of a certain linear system. The columns of the matrix of the system are $\mathbf{u}_1$ and $\mathbf{u}_2$, and it's augmented by $\mathbf{v}$. If one solves the system, one obtains the scalars α_1 and α_2.

Solution:

a) We row reduce the augmented matrix of the system that answers the question:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_2 - R_1,\ R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right].$$

We see that the last row corresponds to the equation $0 = 2$, hence the system is not consistent. In terms of vectors, this means that $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

b) This is almost the same problem except we have one extra vector at our disposal. Row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_2 - R_1,\ R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

The system is consistent, hence $\mathbf{v}$ *is* a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. To find the actual coefficients, we proceed with back substitution:

$$\alpha_3 = 1, \quad \rightarrow \quad \alpha_2 = 1 + \alpha_3 = 2, \quad \rightarrow \quad \alpha_1 = 1 - \alpha_3 = 0.$$

Thus $\mathbf{v} = 2\mathbf{u}_2 + \mathbf{u}_3$.

4. Given the augmented matrix A of some linear system below, describe how many solutions the system has depending on the values of h and k:

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 2 & -1 & -2 & k \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Hint: first, row reduce the matrix to find its REF and determine its rank.

Solution: Row reduce the matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 2 & -1 & -2 & k \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2-2R_1,\ R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 0 & 1 & -2 & k-2h \\ 0 & 0 & 3 & 1-k+2h \end{array}\right]$$

We see that the rank of the system is three and the number of variables is three. When these two numbers coincide, the system is consistent for any possible augmentation (right-hand side of the system), and the solution is unique.

5. Use Gauss-Jordan algorithm to solve the linear system given by its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 1 & 3 & 0 & -2 & 1 \\ 1 & -2 & -5 & 3 & 1 \\ 1 & 2 & -1 & -1 & 1 \end{array}\right]$$

Solution: Row reduce the matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 1 & 3 & 0 & -2 & 1 \\ 1 & -2 & -5 & 3 & 1 \\ 1 & 2 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_2-R_1,\ R_3-R_1,\ R_4-R_1} \left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 0 & 4 & 4 & -4 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 3 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 4 & 4 & -4 & 0 \\ 0 & 3 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_3+4R_2,\ R_4+3R_2} \left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -4 & 2 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is in the reduced row echelon form. Now, the rank is 2, hence there are two free variables. Let $x_3 := t$ and $x_4 := s$ be parameters. Then we solve for the leading variables in terms of t and s:

$$\begin{cases} x_1 - 3t + s = 1 \\ x_2 + t - s = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 + 3t - s \\ x_2 = -t + s \end{cases}$$

Therefore, the general solution assumes the following vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 + 3t - s \\ -t + s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$